Characterization of Compact Metric Spaces

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Abstract

In this article, we offer a proof of the main theorem (Theorem ??). The proof here is different from the one given in my book "Topology of Metric Spaces" [1] and the one in another article of mine with the same title [2] as this one. I believe that this proof is somewhat easier for students to understand and retain than the one in the book, which is perhaps a little more demanding.

We assume that the reader is acquainted with the concepts such as compact spaces in terms of open covers, completeness of a metric space, total boundedness of a metric space and cluster points of a set in a metric space.

Theorem 1. *Let (X, d) be a metric space. The following are equivalent.*

(i) *If $E \subset X$ is an infinite subset of X, then there exists $x \in X$ such that x is a cluster point of E.*

(ii) *Any sequence in X has a convergent subsequence.*

Proof. (i) $\implies$ (ii). Let (x_n) be a sequence in X. If the set $\{x_n : n \in \mathbb{N}\}$ is finite, there exists $x \in X$ such that $S := \{n \in \mathbb{N} : x_n = x\}$ is infinite. (Why?) Since $S \subset \mathbb{N}$ is an infinite set, by the well-ordering principle, we can write $S := \{n_1 < n_2 < \cdots\}$. Hence the subsequence (x_{n_k}) is a constant sequence and hence is convergent.

If $E := \{x_n : n \in \mathbb{N}\}$ is infinite, then there exists $x \in X$ such that x is a cluster point of E. The set $B(x, 1) \cap E$ is infinite. (Why?) Let n_1 be such that $x_{n_1} \in B(x, 1) \cap E$. Now look at $B(x, 1/2) \cap E$. It is an infinite set. Hence there exists $n_2 > n_1$ such that $x_{n_2} \in B(x, 1/2)$. By induction, we choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in B(x, \frac{1}{k+1}) \cap E$. Then (x_{n_k}) is a subsequence which converges to x.

(ii) $\implies$ (i). We need to prove that a cluster point exists for any infinite subset $E \subset X$. Let $E \subset X$ be infinite. We then choose a sequence (x_n) (of distinct elements) in E. (That is, $x_n = x_m$ iff $n = m$). By hypothesis, (x_n) has a convergent subsequence, say, $x_{n_k} \to x$. Then x is a cluster point of E. (Why?) □

Theorem 2. *Let (X, d) be a metric space. The following are equivalent.*

(i) *X is compact.*

(ii) *If $E \subset X$ is an infinite subset of X, then there exists $x \in X$ such that x is a cluster point of E.*

(iii) *(X, d) is complete and totally bounded.*

Proof. (i) $\implies$ (ii): Let E be an infinite subset of X. Assume that there is no cluster point of E in X. What does this mean? Given $x \in X$, x is not a cluster point of E. Hence there exists $r_x > 0$ such that $B'(x, r_x) \cap E = \emptyset$. (As is customary, $B'(x, r) = B(x, r) \setminus \{x\}$.) What can we say of the collection $\{B(x, r_x) : x \in X\}$? It is an open cover of X. What do we know of $B(x, r_x) \cap E$? We see that $B(x, r_x) \subset \{x\}$. Since X is compact, there exists a finite set $F \subset X$ such that $\{B(x, r_x) : x \in F\}$ is a finite subcover. But then we observe

$$E = E \cap X = \cup_{x \in F}(B(x, r_x) \cap E) \subseteq F.$$

That is, E is finite, a contradiction. Hence we conclude that there is $x \in X$ such that x is a cluster point of E.

(ii) $\implies$ (iii): Assume that E is not totally bounded. Then there exists $\varepsilon > 0$ such that there is no ε-net, that is, for no finite subset $F \subset X$, we can have $X = \cup_{x \in F} B(x, \varepsilon)$. Let $x_1 \in X$ be arbitrary. Then $B(x_1, \varepsilon) \neq X$ (Why?) and hence there exists $x_2 \in X$ such that $d(x_2, x_1) \geq \varepsilon$. Assume that we have chosen $x_1, \dots, x_n$ such that $x_k \notin B(x_1, \varepsilon) \cup \cdots \cup B(x_{k-1}, \varepsilon)$ for $2 \leq k \leq n$. Since there exists no ε-net, we know that $X \neq \cup_{k=1}^n B(x_k, \varepsilon)$. Hence there exists $x_{n+1} \in X \setminus (\cup_{k=1}^n B(x_k, \varepsilon))$. Note that $d(x_{n+1}, x_k) \geq \varepsilon$ for $1 \leq k \leq n$. By construction $E := \{x_n : n \in \mathbb{N}\}$ is an infinite set. (Why? If $m < n$), note that $x_n \notin \cup_{k=1}^n B(x_k, \varepsilon)$ implies $d(x_n, x_m) \geq \varepsilon$.) We claim that there is no cluster point for E. If not, let $x \in X$ be a cluster point of E. Look at $B(x, \varepsilon/3) \cap E$. How many points could be there? At most one. Why? If $x_n, x_m \in B(x, \varepsilon/3) \cap E$ with $n \neq m$, then $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) \leq (2/3)\varepsilon$, a contradiction. (Why is this a contradiction?) Hence we conclude that X is totally bounded.

We now show that X is complete. If (x_n) is a Cauchy sequence, then by Theorem 1, it has a convergent subsequence, say, $x_{n_k} \to x$. But it is well-known that $x_n \to x$. So, we have shown that (X, d) is totally bounded and complete.

(iii) $\implies$ (i): The proof is analogous to that of the Heine-Borel theorem which uses the bisection-method and the nested interval theorem. (See [3].) (It is also analogous to the proofs of Cantor's intersection theorem and Baire's theorem, as explained in my book. Do not worry if you have not seen them or do not remember.)

Assume that X is not compact. Let $\{U_i : i \in I\}$ be an open cover from which we cannot extract a finite subcover.

Let $n \in \mathbb{N}$. Since X is totally bounded, there exists a finite set $F_n \subset X$ such that $X = \cup_{x \in F_n} B(x, 1/n)$.

Note that $\{U_i : i \in I\}$ is an open cover of $B[x, 1]$ for any $x \in F_1$. Hence there exists at least one $x_1 \in F_1$ such that we cannot find a finite subcover for $B[x_1, 1]$.

> Why? If no such $x \in F_1$ exists, then each $B[x, 1]$ admits a finite subcover and hence $X = \cup_{x \in F_1} B[x, 1]$, a finite union, will also admit a finite subcover.

We now repeat the argument of the last paragraph replacing X by $K_1 := B[x_1, 1]$ and F_1 by F_2. We have $K_1 = \cup_{x \in F_2}(K_1 \cap B[x, 1/2])$. Hence there exists $x_2 \in F_2$ such that $K_2 := K_1 \cap B[x_2, 1/2]$ does not admit a finite subcover. We proceed by induction to arrive at a sequence (x_n), where $x_n \in F_n$, such that $K_n := K_{n-1} \cap B[x_n, 1/n]$ does not admit a finite subcover. Note that (K_n) is a nested sequence of non-empty sets. The diameter of K_n is at most $2/n$.

Let us choose $y_n \in K_n$, $n \in \mathbb{N}$. Then we claim (y_n) is Cauchy. For, note that if $N \in \mathbb{N}$ is fixed, then for $n \geq N$, we have $y_n \in K_N$. Since $\operatorname{diam}(K_N) \leq 2/N$, it follows that $d(y_n, y_m) \leq 2/N$ for $n, m \geq N$. Let $\varepsilon > 0$ be given. Then by the Archimedean property, there exists $N \in \mathbb{N}$ such that $2/N < \varepsilon$. Thus for $m, n \geq N$, we have $d(y_n, y_m) < \varepsilon$. Since X is complete, there exists $y \in X$ such that $y_n \to y$. Fix $j \in I$ such that $y \in U_j$. Since U_j is open, there exists $r > 0$ such that $B(y,r) \subset U_j$. Since $y_n \to y$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, we have $y_n \in B(y,r) \subset U_j$. Let $n \geq N$ and $z \in K_n$. Then

$$d(z,y) \leq d(z,y_n) + f(y_n,y) < 2/n + r/2.$$

If we choose $n \geq N$ such that $2/N < r/2$, we see that $K_n \subset B(y,r) \subset U_j$. That is, we have found a finite subcover for K_n. This contradicts the way K_n's are constructed. Hence we conclude that we can extract a finite subcover of X from the open cover $\{U_i : i \in I\}$. ☐

Theorem 3. *Let (X,d) be a metric space.Then the following are equivalent.*
(i) *X is compact.*
(ii) *If E is an infinite subset of X, then there exists $x \in X$ such that x is a cluster point of E.*
(iii) *(X,d) is complete and totally bounded.*
(iv) *Every sequence in X has a convergent subsequence.*

Proof. The part (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i)is proved in Theorem 2. The equivalence (ii) $\iff$ (iv) is proved in Theorem 1. ☐

Remark 4. One may also prove the last theorem via the implications (i) $\implies$ (ii) $\implies$ (iv) $\implies$ (iii) $\implies$ (i).

The proof of (iv) $\implies$ (iii) is 'almost the same' as the proof of (ii) $\implies$ (iii) which is in Theorem 2. The set E in that proof is a sequence which has no convergent subsequence.

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References

[1] S Kumaresan, *Topology of Metric Spaces*, 2nd Revised Edition, Narosa Publishing House (2011).

[2] S Kumaresan, Compact Metric Spaces.

[3] S Kumaresan, Nested Interval Theorem and Its Applications.

The last two items are expository articles. They may be downloaded from https://4dspace.mtts.org.in/ea/ You may use the search facility to locate the articles.

As a ready and quick reference for teachers, we give an outline as a series of graded exercises. In fact, most of the hints are unnecessary for many experienced teachers.

Ex. 5. Let (X, d) be a compact metric space.

1. If every infinite subset of X has a cluster point, then every sequence in X has a convergent subsequence.

   Hint: If (x_n) is a sequence in X, consider two cases: $E := \{x_n : n \in \mathbb{N}\}$ is finite or infinite.

2. If every sequence in X has a convergent subsequence, then any infinite subset of X has a cluster point.

   Hint: If E is an infinite set, consider a sequence (x_n) (in E) of distinct points.

3. Let X be compact. Then any infinite set E has a cluster point.

   Hint: If false, for each $x \in X$, choose $r_x > 0$ such that $B'(x, r_x) \cap E = \emptyset$. Then $\{B(x, r_x) : x \in X\}$ is an open cover which admits no finite subcover. If it does, then conclude E is a finite set.

4. If every infinite subset of X has a cluster point, then X is complete and totally bounded.

   Hint: Completeness follows from Item 1. If X is not totally bounded, then for some $\varepsilon > 0$, there is no ε-net. Use this fact to construct an infinite set $E := \{x_n : n \in \mathbb{N}\}$ such that $d(x_n, x_m) \geq \varepsilon$ for $m \neq n$.

5. If X is totally bounded and complete, then X is compact.

   Hint: The proof is analogous to that of the Heine-Borel theorem which uses the bisection-method and the nested interval theorem. If false, let $\{U_i : i \in I\}$ be an open cover with no finite subcover. Use total boundedness to find $x_1 \in X$ such that $B(x_1, 1)$ has no finite subcover. Let $K_1 := B[x_1, 1]$. By induction, construct $K_n := K_{n-1} \cap B[x_n, 1/n]$ has no finite subcover. Note that (K_n) is nested and the diameters go to 0. Choose $y_n \in K_n$. Then $y_n \to y$. Let $y \in U_j$. For $n \gg 0$, $K_n \subset B(y, r) \subset U_j$.

6. Conclude that the following are equivalent for a metric space X:

   (i) X is compact.

   (ii) If $E \subset X$ is an infinite subset of X, then there exists $x \in X$ such that x is a cluster point of E.

   (iii) (X, d) is complete and totally bounded.

   (iv) Every sequence in X has a convergent subsequence.

   Hint: Note that Items 3–5 show that (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i). Items 1 & 2 establish the equivalence of (ii) and (iv).