L'Hospital's Rules

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L'Hospital's rules deal with limits of the form $\lim_{x\to a} \frac{f(x)}{g(x)}$ when we know (i) $\lim_{x\to a} f(x) = l = \lim_{x\to a} g(x)$ where $\ell = 0$ or $\ell = \infty$ and (ii) $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$. Here $\lim_{x\to a}$ means any of the following limits: $\lim_{x\to a_+}$, $\lim_{x\to a_-}$, $\lim_{x\to\infty}$, $\lim_{x\to-\infty}$ and of course $\lim_{x\to a}$. Also, L may be a real number or $L = \pm\infty$.

We shall prove three major versions of L'Hospital's rules. The other cases are treated in a similar fashion.

Theorem 1 (L'Hospital - 1)**.** *Let J be an open bounded interval. Let $a \in J$. Assume the following:*

(i) *f and g are continuous on $J \setminus \{a\}$.*
(ii) *$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$.*
(iii) *f and g are differentiable on $J \setminus \{a\}$.*
(iv) *The function $x \mapsto g(x)g'(x) \neq 0$ for $x \in J$, $x \neq a$.*
(v) *$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$.*

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$.

Proof. If we set $f(a) = 0$ and $g(a) = 0$, then f and g are continuous at a. We now use the Cauchy mean value theorem below.

$$\begin{aligned} \frac{f(x)}{g(x)} &= \frac{f(x) - f(a)}{g(x) - g(a)} \\ &= \frac{f'(t)}{g'(t)}, \qquad \text{for some } t \text{ between } x \text{ and } a. \end{aligned} \tag{1}$$

(Note that $t = t_x$ depends on x.) To prove the result, let $\varepsilon > 0$ be given. Since $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \implies \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon.$$

Hence if $0 < |x - a| < \delta$, then for any t between x and a, we have $|t - a| < \delta$. Thus (1) shows that for $0 < |x - a| < \delta$, we have

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'(t)}{g'(t)} - L \right| < \varepsilon.$$

The result follows. □

Exercise 2. Can the proof of Theorem 1 be adapted to deal with the case $L = \infty$?

Exercise 3. Adapt the proof of Theorem 1 to deal with one sided limits.

Theorem 4 (L'Hospital - 2)**.** *Let J be an interval of the form (R, ∞). Assume the following:*
(i) *f and g are continuous on J.*
(ii) $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to\infty} g(x)$.
(iii) *f and g are differentiable on J.*
(iv) *The function $x \mapsto g(x)g'(x) \neq 0$ for $x \in J$.*
(v) $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$.
Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$.

Proof. We do the obvious trick. Observe that $t \to 0_+$ iff $1/t \to \infty$. So, we define

$$F(t) := f(1/t) \text{ and } G(t) := g(1/t).$$

Observe that $\lim_{t\to 0_+} F(t) = \lim_{x\to\infty} f(x)$ and that $\lim_{t\to 0_+} G(t) = \lim_{x\to\infty} g(x)$. (Why? Do you see we use a result on the theory of limits here? Can you say what it is and how it is used?)

We now wish to use the one-sided limits version of the last theorem. We have $F'(t) = -\frac{1}{t^2} f'(t)$ and $G'(t) = -\frac{1}{t^2} g'(t)$. Hence by the last theorem we have

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to 0_+} \frac{F(t)}{G(t)} = \lim_{t\to 0_+} \frac{F'(t)}{G'(t)} = \lim_{t\to 0_+} \frac{-t^{-2} f'(1/t)}{-t^{-2} g'(1/t)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L.$$

(Did we use any result from the theory of limits in the last equality above?) This completes the proof. □

Exercise 5. Adapt the last proof for the case $\lim_{x\to-\infty} \frac{f(x)}{g(x)}$.

Theorem 6 (L'Hospital-3)**.** *Let $f, g \colon (R, \infty) \to \mathbb{R}$ be differentiable. Assume the following:*
(i) $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x)$.
(ii) $g'(x) \neq 0$ *for* $x > R$.
(iii) $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L$.
Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$.

Proof. This is perhaps the most demanding of all! Let $\varepsilon > 0$ be given. We may assume $0 < \varepsilon < 2$. Let $R_1 > R$ be such that for $x > R_1$, we have $\left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon/2$.

Let $R_2 > R_1$ be such that for $x > R_2$, $f(x) > 0$ and $g(x) > 0$. Let $R_3 > R_2$ be such that for $x > R_2$, we have $f(x) > f(R_2)$ and $g(x) > g(R_2)$. By Cauchy's mean value theorem, we have, for $R_3 < c < x$,

$$\begin{aligned} \frac{f'(c)}{g'(c)} &= \frac{f(x) - f(R_2)}{g(x) - g(R_2)} \\ &= \frac{f(x)\left(1 - \frac{f(R_2)}{f(x)}\right)}{g(x)\left(1 - \frac{g(R_2)}{g(x)}\right)} \\ &= \frac{f(x)}{g(x)} \psi(x), \quad \text{say.} \end{aligned} \tag{2}$$

(Note that thanks to our assumption $x > R_3$, $0 < \left(1 - \frac{f(R_2)}{f(x)}\right) < 1$. Similar estimate holds for f replaced by g.) The equality (2) says that $\frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}\varphi(x)$, where $\varphi(x) = \frac{1}{\psi(x)}$. Since $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x)$ we see that $\lim_{x\to\infty}\varphi(x) = 1$. Now it should be fairly clear how to proceed.

We need to estimate $\left|\frac{f(x)}{g(x)} - L\right|$. We do the obvious thing now. For $x > R_3$, we have

$$
\begin{aligned}
\left|\frac{f(x)}{g(x)} - L\right| &= \left|\frac{f'(c)}{g'(c)}\varphi(x) - L\right| \\
&= \left|\frac{f'(c)}{g'(c)}(\varphi(x) - 1) + \frac{f'(c)}{g'(c)} - L\right| \\
&\leq \left|\frac{f'(c)}{g'(c)}\right| |\varphi(x) - 1| + \left|\frac{f'(c)}{g'(c)} - L\right|. \qquad (3)
\end{aligned}
$$

Note that $\left|\frac{f'(c)}{g'(c)} - L\right| < \varepsilon/2$ and hence $\left|\frac{f'(c)}{g'(c)}\right| < L + \varepsilon/2 < L + 1$, since $0 < \varepsilon < 2$. We choose $R_4 > R_3$ so that for $x > R_4$, we have $|\varphi(x) - 1| < \frac{\varepsilon}{2(L+1)}$. Thus for $x > R_4$, the first term on the right side of (3) is less than $\varepsilon/2$. Since $R_4 > R_2$, the second terms of (3) is also less than $\varepsilon/2$. The result is proved. □

Exercise 7. This is a meta exercise.
(i) How do you formulate the last result if you wish to deal with (a) $\lim_{x\to-\infty}$, (b) $\lim_{x\to a_+}$ and (c) $\lim_{x\to a_-}$? (It will be helpful if you draw pictures.)
(ii) What will be the analogues of R_j, $1 \leq j \leq 4$ in each of the above cases (a)–(c)?
(iii) Take up one case at a time and walk through the proof of L-Hospital-3 with your choices of R_j and check whether they work.
(iv) How do you deal with the case when $\lim_{x\to a}\frac{f'(x)}{g'(x)} = -\infty$? Here $\lim_{x\to a}$ means any of the following limits $\lim_{x\to a_+}$, $\lim_{x\to a_-}$, $\lim_{x\to\infty}$, $\lim_{x\to-\infty}$ and of course $\lim_{x\to a}$.

This will help you understand the proof of L'Hospital-3 above. This also tells you how to deal with when a book or your teacher says, "The other case is similar".

Exercise 8. Now take any book on Calculus and solve about a dozen problems on indeterminate forms. There is an additional burden on you now: You need to check whether the hypotheses are satisfied when you apply any of these rules!