Grassmann Manifold with Complete Details

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1 Grassmann Manifolds – via Column Reduced Matrices

Let $Gr(k,n) \equiv Gr(k,\mathbb{R}^n)$ denote the set of all k-dimensional vector subspaces of $\mathbb{R}^n$. We wish to make it a differential manifold of dimension $(n-k)k$. Note that as sets $Gr(1,n) = P^n(\mathbb{R})$, the n-dimensional projective space over $\mathbb{R}$.

The first goal is for any given $V \in \mathrm{Gr}(k,\mathbb{R}^n)$, we need find a parametrized set containing V.

Let us find an atlas for $\mathrm{Gr}(k,\mathbb{R}^n)$ using the column-reduced matrices.

Let $V \in \mathrm{Gr}(k,\mathbb{R}^n)$. Let $\{v_1,\dots,v_k\}$ be a basis of (column) vectors for V. We then form a $n \times k$ matrix, say, A_V where the j-th column is the column vector v_j. By its very definition, the rank of A_V is k. Conversely, if A is an $n\times k$ matrix of rank k, then the linear span of the columns is a k-dimensional vector subspace of $\mathbb{R}^n$. Denote it by V_A.

Let $M^k_{n\times k}$ denote the matrices of type $n \times k$ of rank k. Given $A, B \in M^k_{n\times k}$, it can happen $V_A = V_B$, that is, the column spaces of both the matrices are the same. We let $A \sim B$ if $V_A = V_B$. This is an equivalence relation. Let $[A]$ denote the equivalence class of A. When does $V_A = V_B$? This happens precisely when there is a (unique) $T \in GL(k)$ such that $B = AT$.

We hope that you have learnt about the Gauss-Jordan elimination and also of the reduced row echelon forms of a given matrix in your linear algebra course. You might have also seen how they are useful in finding a basis for the row space etc. You may be aware that such a reduced row echelon matrix of a given matrix is unique. Of course similar considerations apply to reduced column echelon matrices. This is what we need below.

Let $A \in M^k_{n\times k}$. Assume for the time being that the first k-rows are linearly independent. That is, $(a_{ij})_{1\le i,j\le k}$ is invertible. Let us recall the fact that given such an $n \times k$ matrix, there exists a unique matrix of the form $\begin{pmatrix} I \\ Z \end{pmatrix}$ where $I = I_{k\times k}$, the identity matrix and Z is an $(n-k)\times k$ matrix, which depends on $[A]$. If $B \sim A$, then A and B have the same column reduced matrix. We refer to the column-reduced matrix as the canonical form of A (or the equivalence class $[A]$). The canonical form is got by column reduction or Gauss-Jordan elimination. It is thus a distinguished representative of the equivalence class $[A]$.

What does this say? If V is represented by A, or equivalently by $[A]$, the matrix Z in the canonical from provides as $(n-k)k$ coordinates of V as V varies (over what?).

Let λ denote an ordered k-tuple of integers satisfying $1 \le \lambda_1 < \cdots < \lambda_k \le n$. Let Λ be the set of such λ's. We say that $A \in M^k_{n\times k}$ is of type λ if the rows $R_{\lambda_1}(A),\dots,R_{\lambda_k}(A)$ are linearly independent. (Here $R_j(A)$ denotes the j-th row of A.) Again by column reduction, there is a canonical form in which the above rows form the identity matrix. The rest of the $(n-k)k$ entries afford us a coordinate system for such matrices. The earlier case is when $\lambda = \{1, 2, \ldots, k\}$.

Let us write this explicitly. Let

$$U_\lambda := \{V \in \mathrm{Gr}(k, \mathbb{R}^n) : \exists A \in M^k_{n\times k} \text{ of type } \lambda \text{ such that } V = V_A\}.$$

Given any such V, let P_A be the canonical form in which the rows $R_{\lambda_1}, \ldots R_{\lambda_k}$ form the identity matrix. Let Z be the $(n-k)\times k$ matrix from of the remaining rows. We let $\varphi_\lambda(V) \equiv \varphi_\lambda(V_A) = Z \in M_{(n-k)\times k}$.

Note that given any $V \in \mathrm{Gr}(k, \mathbb{R}^n)$, there exists λ such that $V \in U_\lambda$. Thus we got hold of a tentative candidate for an atlas on $\mathrm{Gr}(k, \mathbb{R}^n)$, namely, $\{(U_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$. Of course, we need to check the smoothness of transition maps.

Let us look at an example.

Example 1. Let us consider $\mathrm{Gr}(2, \mathbb{R}^3)$. Then $\Lambda = \{(12), (13), (23)\}$. The canonical form of any element in U_{12} is of the form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ r & s \end{pmatrix}$. Hence $\varphi_{12}(V) = (r, s)$.

Let $W \in U_{23}$. The canonical form is a matrix like $\begin{pmatrix} u & v \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence $\varphi_{23}(W) = (u, v)$. If $V \in U_{12} \cap U_{23}$, then, the transition map is $(r, s) \mapsto (u, v)$. So we need to exhibit $u = u(r, s)$ and $v = v(r, s)$ as smooth functions of r and s.

If $V \in U_{12} \cap U_{23}$, then in the canonical form (for U_{12}), we deduce that $r \neq 0$. (Why?) How do we get the canonical form valid in U_{23} from that of U_{12}? Observe

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ r & s \end{pmatrix} \begin{pmatrix} 0 & 1 \\ r & s \end{pmatrix}^{-1} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ r & s \end{pmatrix} \left[\left(\frac{-1}{r} \right) \begin{pmatrix} s & -1 \\ -r & 0 \end{pmatrix} \right] \\ &= \begin{pmatrix} -\frac{s}{r} & \frac{1}{r} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{1}$$

Hence we see that $u = -s/r$ and $v = 1/r$. Thus the transition functions are smooth.

Let $V \in U_\lambda \cap U_\mu$. Let P_λ be the canonical form of A such that $V = V_A$. Let $Q_{\lambda\mu}$ be the submatrix of P_λ consisting of only those rows that are indexed by μ. Let $Z_{\lambda\mu}$ the $(n-k)\times k$ matrix consisting of the remaining rows of P_λ.

Consider the function $f_{\lambda\mu} \colon U_\lambda \to \mathbb{R}$ defined by $f_{\lambda\mu}(V) := \det(Q_{\lambda\mu})$. Then $f_{\lambda\mu}$ is continuous. Also, $V \in U_\lambda \cap U_\mu$ iff $f_{\lambda\mu}(V) \neq 0$. Hence we conclude that $\varphi_\lambda(U_\lambda \cap U_\mu)$ is open.

Let $V \in U_\lambda \cap U_\mu$. Recall that $\varphi_\lambda(V) = Z_\lambda$ and $\varphi_\mu(V) = Z_\mu$. So, we need to ask how to express Z_μ in terms of Z_λ. Clearly, $Z_\mu = Z_{\lambda\mu} Q^{-1}_{\lambda\mu}$. (Note that this is what happened in (1).) This establishes the smoothness of the transition maps.

We need to establish that $\mathrm{Gr}(k, \mathbb{R}^n)$ is Hausdorff. If $V, W \in U_\lambda$, then they can be separated by means of open sets. Let $V \in U_\lambda$ and $W \in U_\mu$, $\lambda \neq \mu$ but they are not in $U_\lambda \cap U_\mu$. Let us understand this. When does $V \in U_\lambda$ but $V \notin U_\mu$? It means that the submatrix $Q_{\lambda\mu}$

is not invertible, that is $f_{\lambda\mu}(V) = 0$. Similar remark applies to W and we conclude that $f_{\mu\lambda}(W) = 0$. This suggests that we employ $f_{\lambda\mu}$ to separate them.

First of all, two observations: (i) $Q_{\lambda\mu} = Q_{\mu\lambda}^{-1}$ on $U_\lambda \cap U_\mu$ and hence (ii) $f_{\lambda\mu} f_{\mu\lambda} = 1$.

Since $V \in U_\lambda \setminus U_\mu$, we see that $f_{\lambda\mu}(V) = 0$. Similarly, $f_{\mu\lambda}(W) = 0$. Let $J := (-1, 1)$. Then $f_{\lambda\mu}^{-1}(J)$ is an open set $\Omega_V \subset U_\lambda$ containing V. Similarly, $f_{\mu\lambda}^{-1}(J)$ is an open set $\Omega_W \subset U_\mu$ containing W. If $Z \in \Omega_V \cap \Omega_W$, then Z is an element of $U_\lambda \cap U_\mu$. Hence $f_{\lambda\mu}(Z)$ makes sense. Since $Z \in \Omega_V$, we conclude that $|f_{\lambda\mu}(Z)| < 1$. Similarly, $|f_{\mu\lambda}(Z)| < 1$. Hence the product $|f_{\lambda\mu}(Z) \cdot f_{\mu\lambda}(Z)| < 1$. This is a contradiction to the observation (ii) above.

2 Grassmann Manifolds – A Linear Algebraic Approach

We discuss the Grasssmann manifolds from a different point of view. This is more demanding and is good training ground for your understanding of linear algebra as well as your perseverance.

Let $Gr(k,n) \equiv Gr(k,\mathbb{R}^n)$ denote the set of all k-dimensional vector subspaces of $\mathbb{R}^n$. We wish to make it a differential manifold of dimension $(n-k)k$. Note that as sets $Gr(1,n) = P^n(\mathbb{R})$, the n-dimensional projective space over $\mathbb{R}$.

The first goal is for any given $V \in \mathrm{Gr}(k,\mathbb{R}^n)$, we need find a parametrized set containing V.

Let V be a k-dimensional linear subspace of $\mathbb{R}^n$. Fix any complementary subspace V' of V such that $\mathbb{R}^n = V \oplus V'$. If we assume that $\mathbb{R}^n$ has an inner product structure, a natural choice of V' is $V^\perp$, the orthogonal complement of V in $\mathbb{R}^n$. We assume that this is the case in the sequel.

Let $\mathbb{U}_V$ be the set of all k-dimensional linear subspaces W such that $W \cap V^\perp = (0)$. Note that $V \in \mathbb{U}_V$.

Let $\pi_V\colon \mathbb{R}^n \to V$ be the orthogonal projection corresponding to the orthogonal decomposition $\mathbb{R}^n = V \oplus V^\perp$.

Claim 2. *The restriction of π_V to $W \in \mathbb{U}_V$ is a linear isomorphism onto V.*

Let $w \in W$. Let $w = v + u$, $v \in V$ and $u \in V^\perp$. Then $\pi_V(w) = v$. If $w \in \ker \pi_V$, then $v = 0$ ad hence $w = u \in W \cap V^\perp = (0)$. Hence π_V restricted to W is one-one and the claim follows. □

Let $A \in L(V,V^\perp)$, the set of linear maps from V to $V^\perp$. It is a vector space of dimension $(n-k)k$. We set up a bijection of $\mathbb{U}_V$ and $L(V,V^\perp)$.

For any $A \in L(V,V^\perp)$, we let $\Gamma(A) := \{v+Av : v \in V\} \subset \mathbb{R}^n$. The set $\Gamma(A)$ is traditionally called the *graph* of A.

Claim 3. *$\Gamma(A)$ is a k-dimensional linear subspace of $\mathbb{R}^n$ such that $\Gamma(A) \cap V^\perp = (0)$.*

That it is a linear subspace is easy. Let $x + Ax \in \Gamma(A) \cap V^\perp$. Since $x + Ax \in V^\perp$ and since $Ax \in V^\perp$ (since A maps V to $V^\perp$), we see that $x = (x + Ax) - Ax \in V^\perp$. Thus $x \in V \cap V^\perp = (0)$. □

Claim 4. *Let $W \in \mathbb{U}_V$. Then there exists a unique $A \in L(V,V^\perp)$ such that $W = \Gamma(A)$.*

Let $x \in W$. Using the decomposition $\mathbb{R}^n = V \oplus V^\perp$, we write $x = y + z$ with $y \in V$ an $z \in V^\perp$. If the claim is true, then $y + z = v + Av$ and hence we are led to define $Ay = z$.

Is A well-defined? That is, if $x = y + z$ and $x_1 = y + z_1$ for $x, x_1 \in W$, we need to ensure that $z = z_1$. We observe that $z - z_1 = x - x_1 \in W$. But then the LHS $z - z_1 \in V^\perp$. Thus the element $z - z_1 \in W \cap V^\perp = (0)$. We therefore conclude that $z = z_1$.

That A is linear is easy.

Is A unique? That is, if $B \in L(V,V^\perp)$ is such that $W = \Gamma(A) = \Gamma(B)$, then is $B = A$? Let $v \in V$. Then $x := v + Av \in W$ as well as $y := v + Bv \in W$. Then $x - y \in W$. But then $x - y = Av - Bv \in V^\perp$. Since $W \cap V^\perp = (0)$, we conclude that $Av = Bv$ for $v \in V$. That is, $A = B$. □

Thus we have established a bijection between $\mathbb{U}_V$ and $L(V,V^\perp)$:

Claim 5. *The map $\varphi_V\colon L(V,V^\perp) \to \mathbb{U}_V$ defined by $\psi_V(A) := \Gamma(A)$ is a bijection.* □

Let $\varphi_V := \psi_V^{-1} \colon \mathbb{U}_V \to L(V, V^\perp)$ be its inverse.

Thus we arrive at a 'plausible' candidate for an atlas on $\mathrm{Gr}(k, \mathbb{R}^n)$. It is the collection $\{(\mathbb{U}_V, \varphi_V) : V \in \mathrm{Gr}(k, \mathbb{R}^n)\}$.

Claim 6. $\varphi(\mathbb{U}_V \cap \mathbb{U}_W) = \{A \in L(V, V^\perp) : \Gamma(A) \cap W^\perp = (0)\}$.

This is clear in view of the last two Claims. □

Claim 7. *$\varphi(\mathbb{U}_V \cap \mathbb{U}_W)$ is open in $L(V, V^\perp)$.*

Note that if A is in $\varphi(\mathbb{U}_V \cap \mathbb{U}_W)$, then $\Gamma(A) \cap W^\perp = (0)$ and hence the orthogonal projection π_V restricted to W is an isomorphism of W onto V. This is an "open condition" and hence the claim. □

We note that the transition map $\varphi_W \circ \varphi_V^{-1} = \varphi_W \circ \psi_V$.

Let $A \in \varphi(\mathbb{U}_V \cap \mathbb{U}_W)$. Let $Z := \psi_V(A) = \Gamma(A)$. Let $B := (\varphi_W \circ \psi_V)(A)$.

Then $B \in L(W, W^\perp)$ is the (unique) element such that $\Gamma(B) = Z$.

It therefore follows that we need to show that B "depends smoothly on" A, that is, to express B in terms of A.

Let $z \in W = \Gamma(A) = \Gamma(B)$. Hence there exists $v \in V$ and $w \in W$ such that $z = v + Av = w + Bw$. We then have $v + Av - w = Bw \in W^\perp$. Let $\pi_W \colon \mathbb{R}^n \to W$ be the orthogonal projection. Then $\pi_W(v + Av - w) = \pi_W(Bw) = 0$, since $Bw \in W^\perp$. Let $I_A := I + A \colon V \to \mathbb{R}^n$ be defined by $I_A(v) = v + Av$. Then we see that

$$\pi_W(v + Av - w) = 0 \implies \pi_W(I_A(v)) = w.$$

What do we know of $\pi_W \circ I_A$?

Claim 8. *$\pi_W \circ I_A \colon V \to W$ is a linear isomorphism,*

For, since $A \in \varphi_V(\mathbb{U}_V \cap \mathbb{U}_W)$, $\pi_V \colon \Gamma(A) \to V$ and $\pi_W \colon \Gamma(A) \to W$ are linear isomorphisms.) □

Hence we can express v as $v = (\pi_W \circ I_A)^{-1}(w)$. Finally,

$$Bw = I_A(v) - w = I_A((\pi_W \circ I_A)^{-1})(w) - w.$$

Claim 9. *B "depends smoothly on" A.* □

Let $F \subset \{1, \ldots, n\}$ be a subset of k elements. Let V_F be the linear span of $\{e_i : i \in F\}$. Let $\mathbb{U}_F := \mathbb{U}_{V_F}$. What does φ_F stand for?

Claim 10. *Given any $V \in Gr(k, n)$ there exists a k-subset $F \subset \{1, \ldots, n\}$ such that $V \in \mathbb{U}_F$. Hence $\{(\mathbb{U}_F, \varphi_F) : F \subset \{1, \ldots, n\}, |F| = k\}$ is a finite atlas for $Gr(k, n)$.* □

Claim 11. *Let $V_j \in Gr(k, \mathbb{R}^n)$, $j = 1, 2$. Then there exists $V \in Gr(k, \mathbb{R}^n)$ such that $V_j \in \mathbb{U}_V$, $j = 1, 2$.*

What are the 'extreme' cases for the pair (V_1, V_2)? It could be they are orthogonal complements of each other (of course, this can happen only when $n = 2k$.) Or their intersection is nontrivial. Experiment. Draw pictures for $Gr(2, 4)$. Look at various possibilities such as $V_1 = \mathrm{span}\,\{e_1, e_2\}$ and $V_2 = \mathrm{span}\,\{e_3, e_4\}$ or $V_1 := \mathrm{span}\,\{e_1, e_2\}$ and $V_2 = \mathrm{span}\,\{e_2, e_3\}$ and arrive at a candidate for V.

Let $\{u_1, \dots, u_r\}$ be an orthonormal basis of $V_1 \cap V_2$. If $V_1 \cap V_2 = (0)$, then the basis is the empty set!

Let $\{u_1, \dots, u_r, v_1, \dots, v_s\}$ be an orthonormal basis for V_1. Let $\{u_1, \dots, u_r, w_1, \dots, w_s\}$ be an orthonormal basis for V_2.

What do you know about $\langle v_i, w_j \rangle$? What is $r + s$?

Claim 12. *Let V be the linear subspace spanned by $\{u_1, \dots, u_r, v_1 + w_1, \dots, v_s + w_s\}$. It is k-dimensional.*

Let $c_1 u_1 + \cdots + c_r u_r + d_1(v_1 + w_1) + \cdots + d_s(v_s + w_s) = 0$. That is,

$$c_1 u_1 + \cdots + c_r u_r + d_1 v_1 + \cdots + d_s v_s + d_1 w_1 + \cdots + d_s w_s = 0.$$

It follows that $c_i = 0$, $1 \le i \le r$ and $d_j = 0$, $1 \le j \le s$.

Claim 13. *$V_j \cap V^\perp = (0)$, $j = 1, 2$. Hence $V_j \in \mathbb{U}_V$.*

Let $x \in V_1 \cap V^\perp$. Then $\langle x, u_i \rangle = 0$ for $1 \le i \le r$, since $u_i \in V$, $1 \le i \le r$. Hence we can writ $x = c_1 v_1 + \cdots + c_s v_s$ as $x \in V_1$. Since $x \in V^\perp$, we have

$$0 = \langle x, v_i + w_i \rangle = \langle x, v_i \rangle + \sum_{j=1}^{s} c_j \langle v_j, v_i + w_i \rangle = \sum_j c_j \delta_{ij} + \sum_j c_j 0 = c_i + 0$$

Hence $x = 0$. □

Claim 14. *$Gr(k, n)$ is Hausdorff.* □

Let $V \le \mathbb{R}^n$ be a k-dimensional vector subspace. Let π_V denote the corresponding orthogonal projection. Note that the "operator norm" of P_V is 1: $\| P_v x \| \le \| x \|$ and for $0 \ne x \in V$, we have $\| P_V x \| = \| x \|$ for any $x \in V$.

It is easy to check that P_V is symmetric: $\langle Px, y \rangle = \langle x, Py \rangle$ for any $x, y \in \mathbb{R}^n$. We have $P_V^2 = P_V$ with $V = \mathrm{Im}\,(P_V)$. Also, $\lambda = 1$ is an eigenvalue of multiplicity k and hence $\mathrm{Tr}(P_V) = k$. This gives a heuristic proof of the following claim.

Claim 15. *The map $V \mapsto P_V$ from $Gr(k, n)$ to $\{P \in L(\mathbb{R}^n) : P^2 = P,\ P^t = P,\ \mathrm{Tr}(P) = k\}$ is a bijection. Hence $Gr(k, n)$ is compact.* □

3 Milnor's Proof

In this section we give Milnor's proof of the following result. The proof uses the change of variable formula. Give Ref!

Theorem 16. *There are no continuously differentiable tangent vector field F with $\| F(p) \| = 1$ for $p \in S^{2k}$.*

We need some preliminary lemmas. Recall that $f\colon (X,d) \to (Y,d)$ is lipschitz if there exists a constant L such that $d(f(x), f(x')) \leq Ld(x,x')$ for all $x, x' \in X$. We say f is locally lipschitz if for every $x \in X$ there exists a neighbourhood U_x of x such that the restriction of f to U_x is lipschitz map from U_x to Y.

Let us reacall the following lemma Give Ref!

Lemma 17. *Let (X,d) be a compact metric space. Let $f\colon X \to Y$ be locally Lipschitz from X into another metric space Y. Then f is Lipschitz on X.*

Proof. By local lipschitz condition, for any $x \in X$ there exist $r_x > 0$ and $L_x > 0$ such that $d(f(x_1), f(x_2)) \leq L_x d(x_1, x_2)$ for all $x_1, x_2 \in B(x, r_x)$. By compactness, there exist finitely many points x_i such that $X = \cup B(x_i, r_i)$ where $r_i := r_{x_i}$. We let L_i stand for the lipschitz constant corresponding to x_i and B_i for $B(x_i, r_i)$. Consider the continuous function $h\colon X \times X \setminus \cup_i (B_i \times B_i)$ given by $h(x,y) := d(x,y)$. Then h is a continuous function on a compact set taking values in positive reals. Hence there exists $\varepsilon > 0$ such that $h(x,y) \geq \varepsilon$ for all (x,y) in the domain of the function h. If we take $M \geq \max\{L_i, \operatorname{diam} f(X)/\varepsilon\}$, then M is a lipschitz constant for f on X. □

Lemma 18. *Let $f\colon U \to \mathbb{R}^m$ be a C^1 map from an open set U in $\mathbb{R}^n$. Let K be a compact set in U. Then $f\colon K \to \mathbb{R}^m$ is Lipschitz.*

Proof. This follows easily from the mean value theorem of differential calculus and the last lemma. By the mean value theorem, if $B[x, r_x] \subset U$, we have

$$\| f(x_1) - f(x_2) \| \leq \sup_{0 \leq t \leq 1} \| Df(x_1 + t(x_2 - x_1)) \| \, \| x_1 - x_2 \| , \qquad x_1, x_2 \in B[x, r_x].$$

Since Df is continuous on U and hence on the compact set $B[x, r_x]$, f is lipschitz with the lipschitz constant $L_x = \sup\{\| Df(z) \| : z \in B[x, r_x]\}$. Thus f is locally lipschitz on K and hence lipschitz on K. □

Lemma 19. *Let U be an open connected bounded set in $\mathbb{R}^n$ so that $A = \overline{U}$ is compact and connected. Let F be a continuously differentiable vector field in an open set $V \supset A$. For $t \in \mathbb{R}$, let $F_t(x) := x + tF(x)$, for $x \in A$. If t is sufficiently small, then the mapping F_t is one-to-one and maps A onto $F_t(A)$ whose volume is a polynomial function of t.*

Explain why $F_t(A)$ is a J-set

Proof. Since A is compact and F is C^1, F is lipschitz on A, say with lipschitz constant L: $\| F(x) - F(y) \| \leq L \| x - y \|$, for $x, y \in A$. If t is such that F_t is not one-to-one, then $F_t(x) = F_t(y)$ so that $x - y = t(F(x) - F(y)$ and hence $\| x - y \| \leq L \, |t| \, \| x - y \|$. So, if we choose $|t| < 1/L$, then F_t is one-to-one. The Jacobian matrix of F_t is of the form $I + t(\frac{\partial f_i}{\partial x_j})$, where I is the identity matrix. Hence the determinant of the Jacobian, DF_t is a polynomial function of t of the form $1+t\alpha_1(x)+\cdots+t^n\alpha_n(x)$ where α_i are continuous functions of x. By change of variable formula, we see that the volume of the image of A under F_t is a polynomial function of t:

$$m(F_t(A)) = a_0 + a_1 t + \cdots + a_n t^n,$$

where a_i is the integral of α_i over A. □

Lemma 20. *Assume that $F\colon S^{n-1} \to \mathbb{R}^n$ be a C^1 tangent vector field on the sphere with $\|F(x)\| = 1$ for all x. If t is sufficiently small, then F_t maps the unit sphere in $\mathbb{R}^n$ onto the sphere of radius $\sqrt{1+t^2}$.*

Proof 1. Assume that A is defined by the inequalities: $1/2 \le \|x\| \le 3/2$. We extend the vector field F on A by setting $F(x) := \|x\| F(x/\|x\|)$. We also define $F_t(x) = x + tF(x)$ on this set A. Choose t small enough so that $|t| < 1/3$ and $t < L^{-1}$. (L is the lipschitz constant of F.) For each $v_0 \in S^{n-1}$, the map $\varphi\colon x \mapsto v_0 - tF(x)$ maps the complete metric space A into itself. φ is a contraction. Hence by contraction mapping theorem there exists a unique fixed point. Consequently, the equation $F_t(x) = v_0$ has a unique solution. Thus for a given $v_0 \in S^{n-1}$, $F_t(x) = v_0$ has a unique solution in A. Multiplying both x and v_0 by $\sqrt{1+t^2}$, the lemma follows. (Note that $F_t(rx) = rF_t(x)$.) □

Proof 2. We assume that $n \ge 2$. If t is sufficiently small, then $DF_t(x)$ is nonsingular on all of the compact set A. (This follows from the expression for the determinant of the Jacobian matrix $DF_t(x)$. See the proof of Lemma 19. Or, observe that the set of invertible matrices is an open set, I lies in the open set and for t near to 0, the Jacobian matrices $DF_t(x)$ all lie in a neighbourhood of I for all $x \in A$.) By inverse mapping theorem, F_t is an open map and hence maps the interior of A into an open subset and $F_t(S^{n-1})$ is a relatively open subset of the sphere of radius $\sqrt{1+t^2}$. But $F_t(S^{n-1})$ is a compact and hence closed subset of the sphere of radius $\sqrt{1+t^2}$. Since $n \ge 2$, the spheres in $\mathbb{R}^n$ are connected. Hence $F_t(S^{n-1})$ is the sphere of radius $\sqrt{1+t^2}$. □

Proof of Thm. 16. Given a C^1 field F of unit tangent vectors on S^{n-1}, we consider any annular region $a \le \|x\| \le b$ and extend F to this region as in the last lemma. Then F_t maps the sphere of radius r *onto* the sphere of radius $r\sqrt{1+t^2}$, for t near 0. Hence F_t maps the region A onto the annular region between the spheres of radii $a\sqrt{1+t^2}$ and $b\sqrt{1+t^2}$. Obviously, the volume of the latter region is given by

$$\text{Volume of } F_t(A) = (\sqrt{1+t^2})^n \text{Volume of } A.$$

If n is odd the volume of $F_t(A)$ is not a polynomial function of t. This contradicts Lemma 19. □

Theorem 21. *An even dimensional sphere does not admit a continuous nowhere vanishing tangent vector field.*

Proof. Suppose F is such vector field. We produce an infinitely differentiable unit tangent vector field. This will contradict Theorem 16.

Let $m := \inf\{\|F(x)\| : x \in S^{n-1}\}$. By (Stone-)Weierstrass theorem there exists a polynomial map $P\colon S^{n-1} \to \mathbb{R}^n$ such that $\|P(x) - F(x)\| < m/2$ for all $x \in S^{n-1}$. We define a differentiable vector field G by setting $G(x) := P(x) - \langle P(x), x\rangle\, x$ for $x \in S$. Then G is tangent to S. Also, G is nowhere zero. Let, if possible, $G(x_0) = 0$. Then

$$P(x_0) = \langle P(x_0), x_0\rangle\, x_0. \tag{2}$$

Since $\| P(x) - F(x) \| < m/2$, by Cauchy-Schwarz inequality

$$|\langle P(x) - F(x), x\rangle| < m/2. \tag{3}$$

But $\langle P(x) - F(x), x\rangle = \langle P(x), x\rangle$, since $\langle F(x), x\rangle = 0$. It the follows from Eq. 3 that

$$|\langle P(x), x\rangle| < m/2. \tag{4}$$

Using this inequality in (2) we get

$$\| P(x_0) \| = |\langle P(x_0), x_0\rangle| \, \| x_0 \| < m/2. \tag{5}$$

Since $\| F(x) \| \geq m$ and $\| F(x) - P(x) \| < m/2$, by triangle inequality we see that $\| P(x) \| \geq m/2$ for all x. This contradicts Eq. 5. Hence there is no x_0 with $G(x_0) = 0$. The vector field $G(x)/ \| G(x) \|$ is then a smooth unit tangent field on S. □

Reference Milnor, J., Analytic Proofs of the “Hairy Ball Theorem” and the Brouwer Fixed Point Theorem, Amer. Math. Monthly, *vol.*85, 1978.