Noncontractibility of the Circle

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The aim of this article is to classify the homotopy classes of maps from a circle to the punctured plane

We prove that the circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ is not contractible and derive its consequences. We start with a lemma from complex analysis which says that it is possible to assign the argument of a complex number in a continuous fashion if we restrict ourselves to $\mathbb{C}$ minus $\{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$, or the complex plane minus any closed half line starting from the origin.

Lemma 1. *There exists a continuous map*

$$\alpha \colon X := C \setminus \{z \in \mathbb{C} : z \in \mathbb{R} \text{ and } \leq 0\} \to (-\pi, \pi)$$

such that $z = |z| e^{i\alpha(z)}$ *for all* $z \in X$.

Proof. Let us define the following open half-planes whose union is X: $H_1 := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$, $H_2 := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ and $H_3 := \{z \in \mathbb{C} : \operatorname{Im} z < 0\}$. We define α_i on H_i which glue together to give the required map.

Let $z \in H_1$. Then $\operatorname{Re} z = |z| \cos\theta$ for some $\theta \in [-\pi, \pi]$ and hence $\cos\theta > 0$. This means that $\theta \in (-\pi/2, \pi/2)$. sin is increasing on $(-\pi/2, \pi/2)$ so that we have the continuous inverse $\sin^{-1} \colon (-1, 1) \to (-\pi/2, \pi/2)$. We define $\alpha_1(z) := \sin^{-1}\left(\frac{\operatorname{Im} z}{|z|}\right)$. We can similarly define $\alpha_2 \colon H_2 \to (0, \pi)$ and $\alpha_3 \colon H_3 \to (-\pi, 0)$ by

$$\begin{aligned} \alpha_2(z) &= \cos^{-1}\left(\frac{\operatorname{Re} z}{|z|}\right) \\ \alpha_3(z) &= \cos^{-1}\left(\frac{\operatorname{Re} z}{|z|}\right). \end{aligned}$$

One easily sees that they agree upon their common domains. Thus we get the required function α. □

Definition 2. Let f and g be continuous functions from a space X to Y. Then f and g are *homotopic* iff there is a continuous function $H \colon I \times X \to Y$ such that $H(0, x) = f(x)$ and $H(1, x) = g(x)$ for all $x \in X$. H is called a homotopy from f to g. Thus a homotopy enables one to *pass continuously* from one map to another.

Lemma 3. *Assume that $f\colon S^1 \to S^1$ is homotopic to a constant map. Then there is a continuous function $\varphi\colon S^1 \to \mathbb{R}$ such that $f(x) = e^{i\varphi(x)}$ for all $x \in S^1$.*

Proof. Let $H\colon I \times S^1 \to S^1$ be a homotopy with $H(0,x) = c$ and $H(1,x) = f(x)$ for $x \in S^1$. Since H is uniformly continuous, for $\varepsilon = 2$, there is a $\delta > 0$ such that

$$|H(s,x) - H(t,x)| < 2, \qquad \text{for } |s-t| < \delta, \quad x \in S^1.$$

Let $0 = t_0 < t_1 \cdots < t_n = 1$ be a partition of I such that $|t_i - t_{i+1}| < \delta$ for $0 \le i \le n-1$. Note that $H(0,x) = c = e^{i\psi(x)}$ for some constant map $\psi\colon S^1 \to \mathbb{R}$. We show that $H(t_1,x) = e^{i\varphi_1(x)}$ for some φ_1.

Since $|H(t_1,x) - H(0,x)| < 2$, we see that $H(t_1,x) \neq -H(0,x)$ and hence that $\frac{H(t_1,x)}{H(0,x)} \neq -1$ for $x \in S^1$. We define a continuous function $\alpha\colon S^1 \to \mathbb{R}$ by setting $\alpha(x)$ to be the argument of x taking values in $(-\pi,\pi)$. (This is possible by Lemma 1.) Thus $\frac{H(t_1,x)}{H(0,x)} = e^{i\alpha(x)}$ and consequently

$$H(t_1,x) = e^{i\alpha(x)}H(0,x) = e^{i(\psi(x)+\alpha(x))} = e^{i\varphi_1(x)},$$

where $\varphi_1(x) = \psi(x) + \alpha(x)$. Continuing this way proves the lemma. □

Definition 4. A space is said to be *contractible* if there is a homotopy between the identity map and a constant map.

Ex. 5. Any convex subset of $\mathbb{R}^n$ is contractible.

Theorem 6. *The circle S^1 is not contractible.*

Proof. If it were, then by Lemma 3 there is a function $\varphi\colon S^1 \to \mathbb{R}$ such that $Id(x) \equiv x = e^{i\varphi(x)}$ for all $x \in S^1$. Hence φ is 1-1 and in particular $\varphi(x) \neq \varphi(-x)$. Define $g\colon S^1 \to \{\pm 1\}$ by

$$g(x) := \frac{\varphi(x) - \varphi(-x)}{|\varphi(x) - \varphi(-x)|}.$$

Then g maps S^1 continuously *onto* $\{\pm 1\}$. This contradicts the connectedness of S^1. □

Definition 7. A subset A of a space X is a *retract* of X if there is a continuous function $r\colon X \to A$ such that $r(a) = a$ for all $a \in A$. r is called a retraction of X onto A.

Corollary 8. *There is no retraction of $\mathbb{R}^2$ onto S^1.*

Proof. Let $r\colon \mathbb{R}^2 \to S^1$ be retraction. Let $p = (0,0)$. Define a homotopy $H\colon I \times S^1 \to \mathbb{R}^2$ by $H(t,x) = tp + (1-t)x$. Then $r \circ H\colon I \times S^1 \to S^1$ is a contraction — contradicting Thm. 6. □

Corollary 9 (Brouwer Fixed Point Theorem)**.** *Let $f\colon B[0,1] \to B[0,1]$ be a continuous map. Then f has a fixed point, i.e., there is an $x \in B[0,1]$ such that $f(x) = x$.*

Proof. If there is no point x such that $f(x) = x$, then the two distinct points $f(x)$ and x determine a line joining $f(x)$ and x. We let $g(x)$ be the point on the boundary at which the line starting from $f(x)$ and going to x meets S^1. Then g is a retraction of $B[0,1]$ onto S^1—a contradiction to Corollary 8. In analytical terms, we have $g(x) = x + tv$, where $v = \frac{x - f(x)}{\|x - f(x)\|}$ and $t = -\langle x, v\rangle + \sqrt{1 - \|x\|^2 + (\langle x, v\rangle)^2}$. □

Corollary 10 (Generalised Brouwer Fixed Point Theorem)**.** *Let $f\colon B[0,1] \to \mathbb{R}^2$ be continuous such that $f(S^1) \subset B[0,1]$. Then f has a fixed point.*

Proof. Define $r\colon \mathbb{R}^2 \setminus \{(0,0)\} \to S^1$ by $r(x) = x/|x|$. If $f(x) \neq x$ for all $x \in B(0,1)$ then S^1 can be contracted via the homotopy

$$H(t,x) = \begin{cases} r(x - 2tf(x)), & 0 \leq t \leq 1/2, \\ r((2-2t)x - f((2-2t)x)), & 1/2 \leq t \leq 1. \end{cases}$$

This contradicts Thm. 6. □