Computation of Some Determinants

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December 24, 2023

Abstract

The purpose of this article is to highlight how the characterizing properties of determinants allow us to compute the determinants. Though the determinants are defined using normalization, mutli-linearity and skew-symmetry, students are not exposed to the utility of this definition in the computation of the determinants. Typically, students use the Laplace expansion when evaluating a determinant. We give a few examples to show how the abstract definition coupled/combined with Laplace expansion makes the computations easier. One should also note that Laplace expansion is computationally very inefficient, as it involves (asymptotically) $n!$ operations whereas row operations involve n^3 operations.

Let $\mathbb{F}$ be a field (or a commutative ring with 1). We state the definition, the effect of elementary row/column operations on the determinant of a matrix, $\det(A) = \det(A^T)$, and the determinant of an upper/lower triangular matrix is the product of its diagonal entries.

Example 1. We compute the determinant of the following matrix. $\begin{pmatrix} 2 & 4 & 5 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 3 & 1 & 2 \\ 6 & -5 & 4 & -3 \end{pmatrix}$. By carrying out the elementary row operations $R_1 - 2R_2$, $R_3 + 2R_2$, $R_4 - 6R_2$, $R_1 \leftrightarrow R_2$, $R_2 - R_3$, $R_3 - 3R_2$, $R_3 + R_4$ and $R_4 + 2R_3$, we arrive at upper triangular matrix

$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & -23 \end{pmatrix}$ whose determinant is -23. Since we employed $R_1 \leftrightarrow R_2$, the determinant of the original matrix is 23.

Example 2. Consider the matrix $A :+ \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & -2 & -2 & -3 \end{pmatrix}$. By carrying out the elementary row operations $R_1 \leftrightarrow R_2$, $R_3 + 2R1$, $R_4 - R_1$, $R_4 + 3R_2$, $\frac{1}{5}R_3$, $R_4 - 3R_3$, we arrive the matrix $B := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. Hence we see that $\det A = (-1) \times 5 \times \det B = -10$.

Example 3. Consider the matrix $A = \begin{vmatrix} 1 & 3 & 1 & 1 \\ 2 & 1 & 5 & 2 \\ 1 & -1 & 2 & 3 \\ 4 & 1 & -3 & 7 \end{vmatrix}$. Carrying out the elementary row operation $R_2 - 2R1$, $R_3 - R_1$, and $R_4 - 4R_1$, we arrive at the matrix $B = \begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & -5 & 3 & 0 \\ 0 & -4 & 1 & 2 \\ 0 & -11 & -7 & 3 \end{pmatrix}$. Hence $\det A = \det B = \det \begin{pmatrix} -5 & 3 & 0 \\ -4 & -1 & 2 \\ -11 & -7 & 3 \end{pmatrix}$. Carrying out the elementary row operation $2R_3 - 3R_2$, we arrive at $C = \begin{pmatrix} -5 & 3 & 0 \\ -4 & -1 & 2 \\ -10 & -17 & 0 \end{pmatrix}$. Expanding by the 23-rd element, we obtain $\det C = (-1)^{2+3}[(-5) \times (-17) - (-10) \times 3] = -115$.

Example 4. Consider $D_n := \begin{vmatrix} 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & \dots & 1 & 2 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 2 \end{vmatrix}$. We observe that $D_1 = 2$, $D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ and $D_3 = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 4$. We claim that $D_n = n + 1$.

Expanding D_n by the first row, we get

$$D_n = 2D_{n-1} - \begin{vmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \dots & 0 & 0 & 0 \\ & & & \vdots & \ddots & \vdots & & \\ 0 & 0 & 0 & 0 & \dots & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 2 \end{vmatrix}.$$

We expand the determinant on the RHS by the *first column* to get the recurrence relation $D_n = 2D_{n-1} - D_{n-2}$. Using induction, we show that $D_n = n + 1$.

Example 5. Evaluate $D_n := \begin{vmatrix} 1 & 2 & 3 & \dots & 0 \\ -1 & 0 & 3 & \dots & n \\ -1 & -2 & 0 & \dots & n \\ \vdots & & & & \\ -1 & -2 & -3 & \dots & 0 \end{vmatrix}$. Hint: Have a look at D_3 and D_4, if nothing strikes you.

Example 6 (Vandermonde Determinant). Let $a_1, \cdots, a_n$ be distinct nonzero elements of $\mathbb{R}$.

Consider the column vectors $(a_1^k, a_2^k, \dots, a_n^k)^T$ for $0 \le k \le n-1$. Consider the determinant

$$D(a_1, \dots, a_n) := \begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \dots & a_n^{n-1} \end{vmatrix}.$$

The determinant D is known as Vandermonde determinant and it is ubiquitous in diverse areas of mathematics. The value of the determinant is $\prod_{j>i}(a_j - a_i)$. You may start with $n = 2$ and $n = 3$ to get acquainted with this determinant. We show by induction that $D(a_1, \dots, a_n) = \prod_{j>i}(a_j - a_i)$.

We have $D_2 = \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix} = b - a$. Conisder $D_3 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$. Caary out the $R_3 - aR_2$ to obtain $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 0 & (b-a)b & (c-a)c \end{vmatrix}$. Now we do $R_2 - aR_1$ to arrive at $\begin{pmatrix} 1 & 1 & 1 \\ 0 & b-a & c-a \end{pmatrix}$. We expand by the first column to get

$$D_3 = 1 \cdot \begin{vmatrix} b-a & c-a \\ (b-a)b & (c-a)c \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b & c \end{vmatrix} = (b-a)(c-a)(c-b).$$

Now to evalauate D_n, consider the following column operations in order:

$$R_n - a_1 R_{n-1} \to R_{n-1} - a_1 R_{n-2} \to \dots \to R_2 - a_1 R_1.$$

The resulting determinant is

$$\begin{aligned} D(a_1, a_2, \dots, a_n) &= \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & (a_2 - a_1) & (a_3 - a_1) & \dots & (a_n - a_1) \\ 0 & (a_2 - a_1)a_2 & (a_3 - a_1)a_3 & \dots & (a_n - a_1)a_n \\ & & \vdots & & \\ 0 & (a_2 - a_1)a_2^{n-2} & (a_3 - a_1)a_3^{n-2} & \dots & (a_n - a_1)a_n^{n-2} \end{vmatrix} \\ &= (a_2 - a_1)(a_3 - a_1) \cdots (a_n - a_1) \begin{pmatrix} 1 & 1 & \dots & 1 \\ a_2 & a_3 & \dots & a_n \\ & \vdots & & \\ a_2^{n-2} & a_3^{n-2} & \dots & a_n^{n-2} \end{pmatrix} \\ &= \prod_{j>1}(a_j - a_1) D(a_2, \dots, a_n) \\ &= \prod_{j>1}(a_j - a_1) \prod_{r>k\ge 2}(a_r - a_k) \\ &= \prod_{j>i}(a_j - a_i). \end{aligned}$$

We expanded by the first column and used the multi-linearity of the determinant in its column arguments and arrived at the second equality in the displayed array of equations.

Ex. 7. If $\alpha_1, \dots, a_n$ are distinct complex numbers then for any complex numbers $b_1, \dots, b_n$ there exists a polynomial P of degree at most $n-1$ uniquely determined by the conditions $P(a_j) = b_j$, $1 \le j \le n$. *Hint:* If $P(X) := c + 0 + c_1 X + \cdots + c_{n-1}X^{n-1}$ is the required polynomial, then $(c_0, c_1, \dots, c_{n-1})^T$ is the unqiue solution of a linear system $D(a_1, \dots, a_n)(x_1, \dots, x_n)^T = (b_1, \dots, b_n)^T$. Here $D(a_1, \dots, a_n)$ denotes the vandermonde matrix (and not the determinant).

Example 8. We show that

$$\begin{vmatrix} 1+a_1 & a_2 & \dots & a_n \\ a_1 & 1+a_2 & \dots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \dots & 1+a_n \end{vmatrix} = 1 + a_1 + a_2 + \cdots + a_n.$$

The given determinant is $\det\Big(e_1 + a_1 v, e_2 + a_2 v, \dots, e_n + a_n v\Big)$, where $v = e_1 + \cdots + e_n$. Let us look at $n = 3$.

$$\begin{aligned} \det(e_1 + a_1 v, e_2 + a_2 v, e_3 + a_3 v) &= \det(e_1, e_2 + a_2 v, e_3 + a_3 v) + \det(a_1 v, e_2 + a_2 v, e_3 + a_3 v) \\ &= [\det(e_1, e_2, e_3) + \det(e_1, a_2 v, e_3) + \det(e_1, e_2, a_3 v)] \\ &\qquad + \det(a_1 v, e_2, e_3) \\ &= (1 + a_2 + a_3) + a_1 = 1 + a_1 + a_2 + a_3. \end{aligned}$$

Now proceed by induction.

Example 9. The matrix of this example is from differential geometry and the computation of its determinant is required while computing the volume element of a (hyper) surface given as a graph. The matrix is

$$\begin{pmatrix} 1+x_1^2 & x_1x_2 & \dots & x_1x_n \\ x_2x_1 & 1+x_2^2 & \dots & x_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_nx_1 & x_nx_2 & \dots & 1+x_n^2 \end{pmatrix}.$$

The trick here is, as in the last case, to realize the ith column vector C_i, which is the vector $e_i + x_i v$, where

$$v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_i x_i e_i \in \mathbb{R}^n.$$

Again, it may be worthwhile to start looking at the cases $n = 2, 3$ and then complete the solution by induction.

Example 10. Let $A \in M(n, \mathbb{F})$. Consider the associated linear map $T_A \colon \mathbb{F}^n \to \mathbb{F}^n$ given by $T_A x = Ax$ where $x \in \mathbb{F}^n$ is considered as a column vector. Let $\{v_1, \dots, v_n\}$ be an eigenbasis of $\mathbb{F}^n$. Assume that $T_A v_j = A v_j = \lambda_j v_j$. Then $\det A = \lambda_1 \cdots \lambda_n$.

Note that by the defintion of the determinant, $\det A$ is the unique element of $\mathbb{F}$ such that for any $u_1, \dots, u_n \in \mathbb{F}^n$, we have

$$\det(Au_1, \dots, Au_n) = \det(A)\det(u_1, \dots, u_n).$$

Apply this with $u_j = v_j$ to get the result.