How to Create New Inner Products on $\mathbb{R}^2$?

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Let $(V, \langle , \rangle)$ be a finite dimensional inner product space over $\mathbb{R}$. Let $\{v_i : 1 \leq i \leq n\}$ be a basis of V. It follows from the definition of an inner product, the map $(x, y) \mapsto \langle x, y\rangle$ is *bilinear*, that is, linear in each of its variables: the maps $x \mapsto \langle x, y_0\rangle$ and $y \mapsto \langle x_0, y\rangle$ rare linear from V to $\mathbb{R}$, where x_0, y_0 are held fixed.

Let $f \colon V \to W$ be a linear map from a vector space V to another, W. Let $\{v_i : 1 \leq i \leq n\}$ be a basis of V. If we know the values of f on the basic elements v_i, then we know how to write down the value of $f(v)$ for any $v \in V$. Let $v := \sum_{i=1}^n x_i v_i$, where $x_i \in \mathbb{R}$. Then $f(v) = \sum_{i=1}^n x_i f(v_i)$. This observation prompts us to ask the question: Is there an analogue for an inner product?

Keep the notation of the first paragraph. Let $v := \sum_{i=1}^n x_i v_i$ and $y := \sum_{j=1}^n y_j v_j$. Then we observe that

$$\begin{aligned}\langle v, w\rangle = \left\langle \sum_i x_i v_i, \sum_j y_j v_j \right\rangle &= \sum_{i=1}^n \left\langle x_i v_i, \sum_{j=1}^n y_j v_j \right\rangle \\ &= \sum_{i=1}^n x_i \left\langle v_i, \sum_{j=1}^n y_j v_j \right\rangle = \sum_{i=1}^n \sum_{j=1}^n x_i y_j \langle v_i, v_j\rangle . \end{aligned} \tag{1}$$

Thus, if we know $\langle v_i, v_j\rangle$, $1 \leq i, j \leq n$, then we know how to find $\langle v, w\rangle$ for $v, w \in V$.

Let $a_{ij} := \langle v_i, v_j\rangle$, $1 \leq i, j \leq n$. We note that $a_{ij} = a_{ji}$ for all i, j. Thus if we define A to be the $n \times n$-square matrix whose (i, j)-th entry is a_{ij}, then A is a real symmetric matrix.

Let us now specialize to $V = \mathbb{R}^n$. Let $v_i = e_i$ be the standard i-th basic vector. We think of vectors in $\mathbb{R}^n$ as column vectors (equivalently, as $n \times 1$ matrices). Let $\langle\ ,\ \rangle$ be a (new) inner product on $\mathbb{R}^n$. It follows from the previous discussion, especially, from (1), that

$$\langle x, y\rangle = \sum_{i,j=1}^n a_{ij} x_i y_j, \quad \text{where } a_{ij} := \langle e_i, e_j\rangle . \tag{2}$$

Recall that the standard dot product on $\mathbb{R}^n$ is given by $x \cdot y = y^t x$, the product of the matrix y^t of type $1 \times n$ with the matrix x of type $n \times 1$. As is the practice, the resulting 1×1-matrix

$y^t x$ is identified with its entry, a real number. Note that

$$\langle x, y\rangle = y^t \cdot Ax = Ax \cdot y. \qquad \text{(Verify!)} \tag{3}$$

Thus the new inner product on $\mathbb{R}^n$ is obtained by using a symmetric matrix and the standard dot product on $\mathbb{R}^n$. Take some time to relish this wonderful finding!

Now one can ask the question: Given a real symmetric square matrix A of size n, is the map $(x, y) \mapsto Ax \cdot y$, as defined in (3), an inner product on $\mathbb{R}^n$? Just for simplicity, let us denote the image of (x, y) under this map by $\langle x, y\rangle$, though it may not be an inner product.

Note that the map is bilinear. (Verify! Use various properties of the matrix operations!) So, we need only check whether it is positive definite, that is, whether $Ax \cdot x \geq 0$ for all $x \in \mathbb{R}^n$ and whether $Ax \cdot x = 0$ iff $x = 0$.

Let us restrict ourselves to $\mathbb{R}^2$. Let $A = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$ be real symmetric matrix. Then it follows form (3) that

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = ax_1x_2 + h(x_2y_1 + x_1y_2) + by_1y_2. \qquad (Verify!) \tag{4}$$

As observed earlier, this map is bilinear. Is it positive definite? Clearly this is false. For example, look at the diagonal 2×2-matrix A with diagonal entries 1 and -1. Then $\langle e_1, e_1\rangle = 1$ whereas $\langle e_2, e_2\rangle = -1$. (Verify this.) Also, if A is the matrix all whose entries are 1, then $\langle e_1 - e_2, e_1 - e_2\rangle = 0$.

It transpires that we need to impose some extra conditions on the symmetric matrix A so that we can ensure the positive definiteness. We play around to find these.

Let $x = e_1$, then $\langle e_1, e_1\rangle = a$ using the expression (4). Hence we conclude that $a > 0$. In a similar way, using $x = e_2$, we find that we need to ensure $b > 0$.

Can h be zero? Yes, it can. For, if $h = 0$, then the right side (4) becomes $ax_1x_2 + by_1y_2$. In particular, if $v = (x, y)^t$, then $\langle v, v\rangle = ax^2 + by^2$. This is always nonnegative since a and b are positive. Also, it is zero iff each of the summands ax^2 and by^2 is zero. It follows that $x = 0 = y$ and $v = 0$. Thus positive definiteness is ensured. Let us summarize: If $a > 0$, $b > 0$ and $h = 0$, then (4) defines an inner product on $\mathbb{R}^2$.

So, we assume that $a > 0$, $b > 0$ and $h \neq 0$. Can the determinant $ab - h^2 = 0$? If you know your linear maps well, you may know that A defines a linear map on $\mathbb{R}^2$, where $v \mapsto Av$. If $\det A = 0$, then this linear map has a nontrivial kernel. Let $v = (x, y)^t$ be a nonzero element in the kernel. Then $\langle v, v\rangle = v^t \cdot Av = v^t \cdot 0 = 0$. Therefore we conclude that $\langle\ ,\rangle$ is not an inner product if $\det A = 0$.

Let us prove this in a different way. If $\det A = 0$, then $ab = h^2$. Since $h \neq 0$, we arrive at $\frac{a}{h} = \frac{h}{b}$. Let $t := \frac{a}{h} = \frac{h}{b}$. Then we find that $a = th$ and $h = bt$. Hence $a = t^2 b$. Hence the matrix $A = \begin{pmatrix} bt^2 & tb \\ tb & b \end{pmatrix}$. Let $v = (-1, t)$. (Note that $Av = 0$). An easy calculation using (4) shows that $\langle v, v\rangle = 0$:

$$\text{“}ax^2 + 2hxy + bt^2\text{”} = bt^2 + 2(tb)(-t) + bt^2 = 0.$$

Note that v is a nonzero vector. Hence we conclude that $\det(A) = ab - h^2 \neq 0$.

Can $\det(A)$ be negative? Let $v = (x, y)^t$ be arbitrary. Then

$$\langle v, v\rangle = ax^2 + 2hxy + by^2 = a\left(x + \frac{h}{a}y\right)^2 + \left(b - \frac{h^2}{a}\right)y^2. \tag{5}$$

Note that if $\det(A) < 0$, then for $y \neq 0$, the second term on the right side of the last equation is negative. Can we make the first term zero? Yes, if we let $x = -\frac{h}{a}$. Hence if $v = \left(-\frac{h}{a}, 1\right)$, then $\langle v, v\rangle = \frac{\det(A)}{a} < 0$. Hence we conclude that $\det(A) > 0$.

Let us summarize our findings. If $A = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$ be real symmetric matrix such that (4) is an inner product on $\mathbb{R}^2$, then (i) $a > 0$ and $b > 0$ and (ii) $\det(A) = ab - h^2 > 0$.

We now prove that these conditions are sufficient to ensure that (4) defines an inner product on $\mathbb{R}^2$.

That the map $(v, w) \mapsto w^t Av$ is bilinear is already noted. So we need only check the positive definiteness. Let us use (5). Each of the terms on the right side of this equation is clearly nonnegative. (Why?) If $\langle v, v\rangle = 0$, then each of the terms is zero. Look at the second term. If it is zero, since $\left(b - \frac{h^2}{a}\right) = \frac{\det(A)}{a} > 0$, it follows that $y = 0$. Then the first term becomes $ax^2 = 0$ Since $a > 0$, we conclude that $x = 0$. Hence $v = (0, 0)$.

Before we state the result let us show that one of the conditions is redundant. If $a > 0$ and $\det(A) > 0$, then it follows $b > 0$. For, $ab - h^2 > 0 \iff ab > h^2$. Since $h^2 \geq 0$, it follows that $ab \geq 0$. Since $a > 0$, we conclude that $b \geq 0$. Can it be zero? If it is then $\det(A) = -h^2$. Since $\det(A) > 0$, this is a contradiction. Hence $b > 0$. Thus we have proved the following result.

Theorem 1. *Let* $A = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$ *be real symmetric matrix such that* $a > 0$ *and* $\det A > 0$. *Then the map*

$$\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) \mapsto \left\langle \begin{pmatrix} (x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle := ax_1x_2 + h(x_2y_1 + x_1y_2) + by_1y_2. \tag{6}$$

defines an inner product on $\mathbb{R}^2$.

Furthermore, any inner product on $\mathbb{R}^2$ *arises this way.* $\square$

Ex. 2. Now that you know the secret, go ahead and create at least two new inner products on $\mathbb{R}^2$. For practice, do verify that they are indeed inner products directly. *Hint:* Revisit the "completing the squares" trick in (5).

Remark 3. This remarks aims to put these in a different perspective. The playground is now calculus of two variables. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable functions. That is, all partial derivatives of order less than or equal to 2 exist and are continuous. For simplicity, let us assume that $\frac{\partial f}{\partial x}(0,) = 0 = \frac{\partial y}{\partial (}0, 0)$. Thus the origin is a critical point of f. Is it a local minimum? The way forward to answer this question is to look at the so-called Hessian matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(0,0) & \frac{\partial^2 f}{\partial x \partial y}(0,0) \\ \frac{\partial^2 f}{\partial y \partial x}(0,0) & \frac{\partial^2 f}{\partial y^2}(0,0) \end{pmatrix}.$$

Note that this matrix is symmetric. (Why?) Now $(0,0)$ is a point of local minimum if the Hessian matrix of f at the origin defines an inner product on $\mathbb{R}^2$. In less intimidating terms(?), the origin is a point of local minimum if the Hessian matrix of f at the origin is positive definite.

The analytical argument hinges on the 2nd order Taylor expansion. For all (x,y) near to to the origin we have

$$\begin{aligned} f(x,y) &= f(0,0) + \frac{\partial f}{\partial x}(0,0) + \frac{\partial f}{\partial y}(0,0) + \frac{\partial^2 f}{\partial x^2}(0,0)x^2 + \frac{\partial^2 f}{\partial x \partial y}(0,0)xy + \frac{\partial^2 f}{\partial y^2}(0,0)y^2 \\ &\qquad + \text{higher order terms} \\ &= f(0,0) + 0 + 0 + ax^2 + 2hxy + by^2 + \text{higher order terms} \\ &= f(0,0) + ax^2 + 2hxy + by^2 + \text{higher order terms}, \end{aligned}$$

where $a = \frac{\partial^2 f}{\partial x^2}(0,0)$, $b = \frac{\partial^2 f}{\partial y^2}(0,0)$ and $h = \frac{\partial^2 f}{\partial x \partial y}(0,0)$. Note that for $(x,y) \neq (0,0)$, the expression $ax^2 + 2hxy + by^2 > 0$.

For (x,y) very near to zero, the higher order terms are negligible. So, $f(x,y) - f(0,0)$ is "approximately equal to" $ax^2 + 2hxy + by^2$ which is positive for all $(x,y) \neq (0,0)$ near the origin. Hence $f(0,0)$ is a local minimum. Of course, in analysis, the last two qualitative statements are proved rigorously.