Transfer of Mathematical Structures Using Bijections

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Abstract

Let $f\colon X \to Y$ be a bijection. Assume that one of X and Y has a mathematical structure (for example, it may be a group structure, vector space structure, metric space structure or a topology). Then we may use f to transfer the structure to the other set. We explain this construction with examples and bring out the significance of "isomorphisms" and demystify the way some esoteric examples are constructed.

Let G and X be sets. Let $f\colon G \to X$ be a bijection. Assume that there is a binary operation $*$ on G which makes G a group. We use this to define a binary operation $\dagger$ on X, show that under this binary operation X becomes a group and the given map $f\colon (G,*) \to (X,\dagger)$ is an isomorphism of groups.

Let $x, y \in X$. Then there exist unique $a, b \in G$ such that $f(a) = x$ and $f(b) = y$. In the sequel, we shall denote this correspondence by the symbol $x \leftrightarrow a$ (under f, if we wish to be pedantic). We define

$$x \dagger y := f(a * b).$$

See Figure 1. What happened is summarized below as a recipe.

> **Recipe:** We pull x and y to G using the map f to obtain a and b and use the binary operation on G to get $a * b$ and 'push' it to X via the map f to get $x \dagger y = f(a * b)$.

Let us explain this construction with an example.

Example 1. Let us consider the additive group $G := (\mathbb{R}, +)$. Let $X := \mathbb{R}$ and the bijection $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(t) := t + 1$. We shall use the symbols r, s, t to denote elements of the domain group. Let x, y, z etc refer to elements of the co-domain X. Let the binary operation on X induced by f be denoted by $\dagger$.

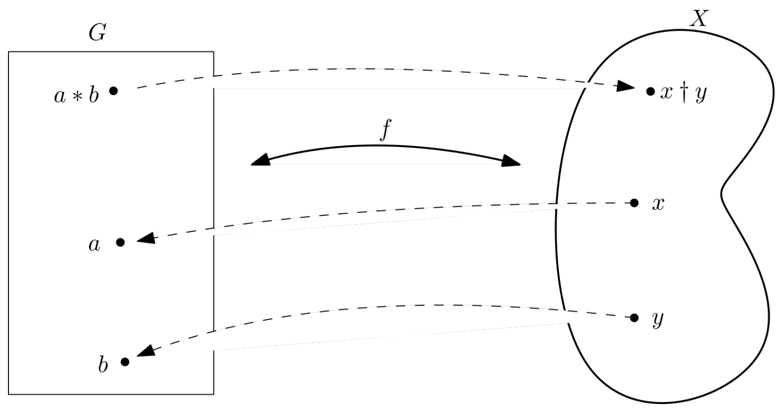

Figure 1: Transfer of Group Structure

Let $x, y \in X$. What is $x \dagger y$? According to our recipe above, we pull x and y to the domain via f. That is, we find r, s in the domain such that $f(r) = x$ and $f(s) = y$. Since $f(r) = r + 1 = x$, we see that $r = x - 1$ and $s = y - 1$. We now carry out the binary operation in the domain group to get $(x-1) + (y-1) = x + y - 2$. We then 'push' $x + y - 2$ via f to the co-domain to get $f(x + y - 2) = x + y - 2 + 1 = x + y - 1$. So we arrive at $x \dagger y = x + y - 1 = x + y + (-1)$ where the $+$ on the right side is the standard addition the numbers x, y and -1.

We leave it to the reader to verify that $(\mathbb{R}, \dagger)$ is a group.

Ex. 2. Let $G = (\mathbb{R}, +)$ and $X = (0, \infty)$, the set of positive reals. Consider G as a group under addition. Let $f \colon G \to X$ be defined by $f(t) := e^t$. What is $x \dagger y$ if $x, y \in X$? Did you get $x \dagger y = x \cdot y$, the product of real numbers?

Let us return to the abstract case. Is the binary operation $\dagger$ on X associative? We use an obvious notation: $x \leftrightarrow r$, $y \leftrightarrow s$ and $z \leftrightarrow t$ under f. Using the definition of $\dagger$ and the fact that f is a bijection we see that $x \dagger (y \dagger z) \leftrightarrow r * (s * t)$ and similarly, $(x \dagger y) \dagger z \leftrightarrow (r * s) * t$. But due to the associativity in G, we know that $r * (s * t) = (r * s) * t$. Again using the bijective nature of f, we conclude that $x \dagger (y \dagger z) = (x \dagger y) \dagger z$.

What will be identity element, say, ϵ of $(X, \dagger)$? An obvious guess is $\epsilon = f(e)$. Let us verify.

$$x \dagger \epsilon = f(r * e) = f(r) = x, \text{ etc.}$$

Can you guess what is the inverse of $x = f(r)$? The inverse of $x \in X$ is $f(r^{-1})$.

Example 3. Let us return to Example 1. The identity of $(\mathbb{R}, \dagger)$ is $f(0) = 1$. Let us verify this.

$$x \dagger 1 = x + 1 - 1 = x,$$

The inverse of $x \in (\mathbb{R}, \dagger)$ is found by the algorithm $x \leftrightarrow x - 1 \mapsto -(x-1) = -x + 1 \leftrightarrow (-x+1) + 1 = -x + 2$. Let us verify this.

$$x \dagger (-x + 2) = x + (-x + 2) - 1 = 1 = f(0).$$

Make sure you understand this. Think also in terms of Figure 1.

Let us summarize our findings.

> We took the additive group $\mathbb{R}$, considered the bijection $f : \mathbb{R} \to X = \mathbb{R}$ given by $f(t) = t + 1$. Using this we transferred the group structure on the domain $\mathbb{R}$ to the co-domain with the binary operation $x \dagger y = x + y - 1$. We found that the 'additive identity' for the binary operation $\dagger$ is 1 and the 'additive inverse' of any $x \in X$ is $-x + 2$.

Ex. 4. Find the identity element and the inverses in the the case of Ex. 2.

Example 5. Let us extend the scope of Example 1. Consider the standard vector space $(\mathbb{R}, +, \cdot)$. Let $X = \mathbb{R}$ and f be as in earlier. We have already understood the induced 'addition $\dagger$ on the co-domain. Let us understand how to transfer the scalar multiplication on the domain to the co-domain X. Let $\lambda \in \mathbb{R}$ be a scalar, $x \in X$. What is $\lambda * x$? x is pulled back to the domain to get $x - 1$. There we carry out the scalar multiplication $\lambda(x - 1) = \lambda x - \lambda$. We then push it to X via the map f to obtain $\lambda x - \lambda + 1$. Let us record it:

$$\lambda * x = \lambda x - \lambda + 1.$$

We need to check whether it satisfies the properties of a scalar multiplication. Let is check the property $1 * x = x$ for $x \in X$, as a sample.

$$1 * x = 1x - 1 + 1 = x.$$

How about $(\alpha\beta) * x = \alpha * (\beta * x)$?

$$(\alpha\beta) * x = \alpha\beta x - \alpha\beta + 1.$$

On the other hand,

$$\begin{aligned} \alpha * (\beta * x) &= \alpha * (\beta x - \beta + 1) \\ &= \alpha(\beta x - \alpha + 1) - \alpha + 1 \\ &= \alpha\beta x - \alpha\beta + \alpha - \alpha + 1 \\ &= \alpha\beta x - \alpha\beta + 1. \end{aligned}$$

Thus we conclude that $(\alpha\beta) * x = \alpha * (\beta * x)$ The other properties are verified similarly.

Let us summarize our findings.

> We used the bijection $f : \mathbb{R} \to \mathbb{R}$ defined by $f(t) := t + 1$ to transfer the vector space structure on the domain to the co-domain with the 'vector addition' $x \dagger y = x + y - 1$ and the 'scalar multiplication': $\lambda * x = \lambda x - \lambda + 1$.

Ex. 6. Keep the notation of Ex 2. Show that X becomes a vector space with the "vector addition" $x \dagger y = x \cdot y$ and the scalar multiplication $\alpha * x = x^\alpha$. Do not forget to verify the properties of the vector addition and the scalar multiplication.

What does the standard basis $\{1\}$ of $\mathbb{R}$ correspond to in X? Call it $\{b\} \subset X$. Given $x \in X$, can you find $\alpha \in \mathbb{R}$ such that $\alpha * b = x$?

Some of you might have seen this example earlier but now you know how this vector space structure on $(0, \infty)$ is arrived at!

Ex. 7. Prove that the maps f in Example 5 and Ex. 6 are isomorphisms.

Ex. 8. Let $f\colon X := \{a, b, c, d, e\} \to \mathbb{Z}_5 := \{\overline{1}, \overline{2}, \dots, \overline{4}, \overline{0}\}$ be the bijection where $f(a) = \overline{1}$, $f(b) = \overline{2}$, etc, and $f(e) = \overline{0}$. Let $\mathbb{Z}_5$ be the cyclic group under addition modulo 5. Equip X with a group structure. Can you say what is the identity element in X? What is the inverse of b? What is the 'addition' of c and d?

Let us jazz up Example 1 a bit. We now use to transfer the multiplication on the domain field $\mathbb{R}$ to the co-domain X. The algorithm goes as follows:

$$x, y \in X \mapsto x - 1, y - 1 \mapsto (x-1)(y-1) = xy - x - y + 1 \mapsto xy - x - y + 2.$$

Hence we define a multiplicative structure on X by the following recipe.

$$x * y = xy - x - y + 2.$$

Remark 9. Beginners should pay attention to the recipes for the scalar multiplication and the multiplication on X are different. To gain mastery, they are urged to review the definitions..

What is the mutiplicative identity? It should be $f(1) = 2 \in X$. Let us verify:

$$x * 2 = 2x - x - 2 + 2 = x.$$

Good, now what is the multiplicative inverse of the "non-zero" $x \in X$? Note that the additive identity is 1 and hence we need to ensure that $x \neq 1$. Thus x is the non-zero element in X. The algorithm says

$$x \leftrightarrow x - 1 \mapsto \frac{1}{x-1} \mapsto f\left(\frac{1}{x-1}\right) = \frac{1}{x-1} + 1 = \frac{x}{x-1}.$$

Let us verify that $\frac{x}{x-1}$ is the 'multiplicative inverse of $1 \neq x \in X$. That is we need to show that $x * \frac{x}{x-1} = 2$.

$$x * \frac{x}{x-1} = x\frac{x}{x-1} - x - \frac{x}{x-1} + 2 = \frac{x^2 - x(x-1) - x + 2(x-1)}{x-1} = 2.$$

The other axioms for a field may be verified in a similar manner to show that $(\mathbb{R}, \dagger, *)$ is a field. Let us summarize our findings.

We took the domain to be the real number field and the co-domain to be $X = \mathbb{R}$. Using the bijection $f\colon \mathbb{R} \to X$ defined by $f(t) = t + 1$, we transferred the field structure on the domain to the co-domain via the following new binary operations on X: $x \dagger y = x + y - 1$ and $x * y := xy - x - y + 2$.

Ex. 10. Let X be a set. Let $f\colon X \to G$ be a bijection where G is a group. How will you equip X with a group structure? Are there analogous question you would like to ask?

Example 11. This kind of "reverse engineering" yields the standard description of $\mathbb{C}$ the field of complex numbers as $\mathbb{R}^2$ with the standard addition and new multiplication.

Let us work with $\mathbb{C}$ as we learned in 'school mathematics'. If $z \in \mathbb{C}$, we write it as $z = x + iy$ where $x, y \in \mathbb{R}$. x (respectively y) is called the real part (respectively, imaginary part) of the complex number z. We say two complex numbers $z = x + iy$ and $z = u + iv$ are equal if $x = u$ and $y = v$. We also define $z + w := (x + u) + i(y + v)$. When we wish to multiply z with w, we expect distributive law and define $z \cdot w = (xy - yv) + i(xv + yu)$.

We now set up a bijection $f\colon \mathbb{C} \to \mathbb{R}^2$ by setting $f(z) = (x, y)$ where $z = x + iy$. Given $(x, y), (u, v) \in \mathbb{R}^2$, we pull them back to $\mathbb{C}$ via f to set $z := x + iy$ and $w := u + iv$ so that $f(z) = (x, y)$ and $f(w) = (u, v)$. We then "define" $(x, y) + (u, v) := f(z + w) = (x + u, y + v)$. We also define the multiplication on $\mathbb{R}^2$ by setting

$$(x, y) \cdot (u, v) := f(z \cdot w) = (xu - yv, xv + yu).$$

With these operations on $\mathbb{R}^2$, one checks that $\mathbb{R}^2$ is a field. You may see this description of $\mathbb{C}$ in a course on complex analysis.

We do not want to elaborate any further. But if you have diligently followed our earlier examples, you can easily work out the details.

Let us turn our attention to metric spaces. Let $f\colon X \to Y$ be a bijection. If one of them has a metric on it, do you know how to define a metric on the other using the bijection?

For example, let (Y, d) be a metric space and $f\colon X \to Y$ be a bijection. Can you transfer the metric d to X? The obvious way to do this to define $d'(x_1, x_2) = d(f(x_1), f(x_2))$ where $x_1, x_2 \in X$. Draw a picture to understand this construction. We leave it to you to check that d' is a metric on X. What can you say of the map $f\colon (X, d) \to (Y, d')$? Do you see immediately that f is an isometry?

Let us look at a concrete case. $X = \mathbb{R}$ with the standard metric and $Y = (-1, 1)$ with the standard metric. I hope you know that $\mathbb{R}$ is complete with the standard metric while $(-1, 1)$ is not complete with the standard metric.

Example 12. Consider $f\colon (-1, 1) \to \mathbb{R}$ defined by $f(t) := \frac{t}{1 - |t|}$. Let the co-domain $\mathbb{R}$ be given the standard metric $d(x, y) := |x - y|$ for $x, y \in \mathbb{R}$. We use f to 'pull-back' the metric d on $\mathbb{R}$ to get a metric d' on $(-1, 1)$ by setting

$$d'(s, t) := d(f(s), f(t)).$$

Since the map $f\colon ((-1,1),d') \to (\mathbb{R},d)$ is an isometry, we conclude that $((-1,1),d')$ is a complete metric space. Just to quell your misgivings, let us check whether the sequence $(n-1/n)$ is Cauchy in d'. We compute

$$\begin{aligned} d'\left(\frac{n-1}{n},\frac{m-1}{m}\right) &= d\left(f\left(\frac{n-1}{n}\right),f\left(\frac{m-1}{m}\right)\right) \\ &= \left|(n-1)-(m-1)\right| = \left|n-m\right|. \end{aligned}$$

Hence we conclude that it is not Cauchy in d' metric. (Do you understand why I took this particular sequence?)

A food for thought: Suppose you are given $((-1,1),d')$ and asked to prove that it is Cauchy. Though it is not difficult, to prove directly using d' will be bewildering due to the nature of d'. This should convince you the significance of recognizing familiar objects in a disguise.

Example 13. Let us consider the inverse $g\colon \mathbb{R} \to (-1,1)$ of f of the last example. We know g is given by $g(x) := \frac{x}{1+|x|}$. Let $(-1,1)$ be given the standard metric d. We then know that the sequence $(\frac{n-1}{n})$ is Cauchy but not convergent in $((-1,1),d)$. We now pull the metric d on $(-1,1)$ to get a metric d' on $\mathbb{R}$. We claim that $(\mathbb{R},d')$ is not complete. Can you think of a sequence which is Cauchy in $(\mathbb{R},d')$ but not convergent in $(\mathbb{R},d')$?

This is a test question to check how much you have absorbed the discussions above! Look for a sequence in the other space which is Cauchy but not convergent and pull it back to $\mathbb{R}$. An obvious choice is $(\frac{n-1}{n})$. The corresponding sequence in $\mathbb{R}$ is $(n-1)$. (How did I write it without any computation?) We leave it to you to carry out the analysis and convince yourself that the sequence is Cauchy but not complete in $(\mathbb{R},d')$.

In Example 12, $(-1,1)$ tries to 'emulate' the complete space $\mathbb{R}$ and it turns to be a complete metric space. In Example 13, $\mathbb{R}$ tries to emulate the incomplete space $(-1,1)$ and it turns to be an incomplete metric space. Is there any life's lesson here?

The last stop is in Topology. Let $f\colon X \to Y$ be a bijection. Assume one of them have a topology $\mathcal{T}$ on it. Can you think of a topology $\mathcal{T}'$ on the other so that f becomes a homeomorphism? A parting gift is the hint: Declare $U \subset X$ is open iff $f(U)$ is open in Y.

I hope that these explorations make you understand the true meaning of 'isomorphisms'. Isomorphisms help us recognize familiar objects in different disguise and labels. Also you are privy to the insider trick of constructing some esoteric examples of groups, vector spaces etc.