Simple and Direct Proofs of Some Results in the Theory of Compact Spaces

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The aim of the is article is provide simple and direct proofs of the following well-known and most useful results in the theory of compact spaces.

Theorem 1 (Heine-Borel). *A subset $K \subset \mathbb{R}^n$ is compact if and only if K is closed and bounded.*

Theorem 2. *Let X be a compact space. Let $f\colon X \to \mathbb{C}$ be continuous. Then*

(i) f is bounded, that is, there exists a positive constant C such that $|f(x)| \leq C$ for all $x \in X$.

(ii) If we further assume that f is real-valued, then f attains its maximum and minimum values.

To make the article self-contained and accessible to students and teachers of B.Sc., I shall include the necessary definitions and prerequisites in the context of metric spaces only. Relevant modifications, if any, for the general case will be indicated at appropriate places. A reader with necessary background may jump to the proofs of these theorems which start after Lemma 12.

Definition 3. A subset K of a metric space (X, d) is said to be *compact* if a family of open sets $\{U_i : i \in I\}$ is given with the property that $K \subseteq \cup_{i\in I} U_i$, then we can find a finite subset $F \in I$, say, $F = \{i_1, i_2, \ldots, i_n\}$ such that $K \subseteq \cup_{i=j}^{n} U_{i_j}$.

We say that the given family is an open cover of K. If we can find such a finite family, we say that we can find/extract a finite subcover for K from the given cover of K.

Thus, K is compact if and only if from any open cover of K, we can extract a finite subcover.

We say that X is compact if X is a compact subset of X.

Example 4. A trivial example is any finite subset $K \subset X$.

Remark 5. The above definition makes sense for any subset of a topological space.

Definition 6. A subset A of a metric space (X, d) is *bounded* if it is contained in an open ball $B(x_0, r)$. for some $x_0 \in X$ and $r > 0$.

Remark 7. Our definition of bounded subsets is very geometric and uses only the primitive notion (that is, the concept of an open ball) in a metric space and is intuitive. By definition the empty set is bounded.

It is an easy exercise to see that if A is bounded if and only if for any arbitrary $x \in X$, there exists $R > 0$ (depending on x) such that $A \subset B(x, R)$. (For, if $A \subset B(x_0, r)$, then $A \subset B(x, R)$ where $R = r + d(x, x_0)$, Check!)

The standard definition runs as follows. Given a **nonempty** set $A \subset X$, we define its *diameter*

$$\text{diam}\,(A) := \sup\{d(x, y) : x, y \in A\}, \text{ as an extended real number, possibly } +\infty.$$

We say that A is bounded if A is empty or if A is nonempty and $\text{diam}\,(A) < \infty$. We leave the equivalence of both the definitions as an easy exercise to the reader.

Lemma 8. *Any compact subset of a metric space is bounded.*

Proof. Let $K \subset X$ be compact. Let $x_0 \in X$ be arbitrary. Consider $U_n := B(x_0, n)$ for $n \in \mathbb{N}$. (Note that $U_i \subseteq U_j$, if $i \leq j$.) We claim that $\{U_n : n \in \mathbb{N}\}$ is an open cover of K. Given any $x \in X$, by the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $d(x, x_0) < n$. Hence $x \in U_n$. As a consequence $X \subset \cup_{n \in \mathbb{N}} U_n$. In particular, $K \subseteq \cup_{n \in \mathbb{N}} U_n$. Since K is compact, there exists a finite subcover, say, $U_{n_1}, \ldots, U_{n_k}$ of K. If we let $N = \max\{n_i : 1 \leq i \leq k\}$, then $U_{n_i} \subset U_N$ so that

$$K \subseteq \cup_{i=1}^{k} U_{n_i} = U_N.$$

That is, $K \subset B(x_0, N)$ and hence bounded. □

Remark 9. As there is no satisfactory notion of bounded subsets in an arbitrary topological space, there is no analogue of the above lemma in the context of topological spaces.

Lemma 10. *Let K be a compact subset of a metric space. Then K is closed in X.*

Proof. It suffices to show that the complement of K in X is open. Let $z \in X \setminus K$ be given. Let $x \in K$ be arbitrary. Then $x \neq z$ and hence there exist open balls $B(x, r_x)$ and $B(z, \delta_x)$ such that $B(x, r_x) \cap B(z, \delta_x) = \emptyset$. The collection $\{B(x, r_x) : x \in K\}$ is obviously an open cover of K. Since K is compact, we can extract a finite subcover, say, $\{B_i = B(x_i, r_{x_i}) : 1 \leq i \leq n\}$. Let us denote by $V_i = B(z, r_{x_i})$ the corresponding open balls centered at z. Then $V := \cap_{i=1}^{n} V_i$ is an open set containing z. We claim that $V \subset X \setminus K$. For, let $x \in K \cap V$. Since B_i form an open cover of K, $x \in B_i$ for some $1 \leq i \leq n$. Now, $x \in B_i \cap V$, but $B_i \cap V \subset B_i \cap V_i = \emptyset$, a contradiction! This proves our claim. We have thus shown any $z \in X \setminus K$ lies in an open set V whose intersection with K is empty. That is, any such z is an interior point of $X \setminus K$. In other words, $X \setminus K$ is open. □

Remark 11. The above result holds true in any Hausdorff space. The open balls $B(x, r_x)$ and $B(z, \delta_x)$ are replaced by open sets $U_x \ni x$ and $V_x \ni z$ such that $U_x \cap V_x = \emptyset$. These are obtained by using the Hausdorff property of the space.

Lemma 12. *If C is closed subset a compact set K in a metric space, then C is compact.*

Proof. First of all observe that C must be closed in X. For, the compact set K is closed in X by the above lemma. Thus C, being a closed subset of a closed set, is closed.

Let $\{U_i : i \in I\}$ be an open cover of C. We add the open set $U := X \setminus C$ to this cover. We claim that $U \cup \{U_i : i \in I\}$ is an open cover of K. For, let $x \in K$ be arbitrary. If $x \in C$, then $x \in \cup_{i \in I} U_i$. If $x \in K \setminus C$, then $x \in U$. Thus, given any $x \in K$, it follows that $x \in U \cup (\cup_{i \in I} U_i)$. The claim follows. Since K is compact, we can extract a finite subcover. Note that U may or may not be a member of this finite subcover. (It may happen that the original cover of C is also an open cover of K.) If U is not a member of this finite cover of K, then C, being a subset of K, is contained in the union of the finite collection. Thus we have extracted a finite subcover of C from the original collection.

In the case when U is a member of the finite subcover of K, we can remove it from the finite collection and we shall be left with a finite collection, say, $\{U_{i_k} : 1 \leq k \leq n\}$. We claim that this is an open cover of C. This is obvious, since

$$C \subset K \subset \left(\cup_{j=1}^{n} U_{i_j} \right) \cup U,$$

given any $x \in C$, x has to lie in one of the U_{i_j}'s only, as $U = X \setminus C$. ☐

Remark 13. The above result remains true in a Hausdorff space. The proof goes through verbatim.

Proof of Theorem 1: Let K be a compact subset of $\mathbb{R}^n$. Then by Lemmas 3 and 4, it follows that K is closed and bounded in $\mathbb{R}^n$.

The difficult part is to prove the converse. Let $K \subset \mathbb{R}^n$ be closed and bounded. Since K is bounded, it follows that $K \subset B(0, R)$ for some $R > 0$. Let Q denote the n-dimensional cube $[-R, R] \times \cdots \times [-R, R]$ (n-times) in $\mathbb{R}^n$. Then $K \subset B(0, R) \subset Q$. (For, if $x = (x_1, \ldots, x_n) \in B(0, R)$, then $\sqrt{\sum_i x_i^2} < R$ so that $|x_i| < R$ for $1 \leq i \leq R$.) If we show that Q is compact, then K, being a closed subset of Q, is compact by Lemma 12.

To make the geometric ideas clear, we shall restrict our attention to the case when $n = 2$. Our proof carries through for all n including $n = 1$.

The crucial geometric observation is contained in the following exercise.

Ex. 14. Let $a, b, c, d \in \mathbb{R}$ be such that $b - a = d - c$. Let $S := [a, b] \times [c, d]$ be the square in $\mathbb{R}^2$. The vertices of S are (a, c), (b, c), (b, d) and (a, d). We call the point (a, c) as the bottom left vertex of S. The pair of midpoints of its opposite sides are given by $([a+b]/2, c)$, $([a+b]/2, d)$ and $(a, [c+d]/2)$, $(b, [c+d]/2])$. By joining the midpoints of opposite sides, we get four smaller squares. Observe that if (a_1, c_1) is the bottom left vertex of any of these squares, we have $a \leq a_1$ and $c \leq c_1$.

Look at Figure 1 to see this geometrically.

We continue with the proof of Heine-Borel Theorem. Let K be a closed and bounded set in $\mathbb{R}^2$. Then there exists $R > 0$ such that $K \subset S := [-R, R] \times [-R, R]$. Since a closed subset of a compact set is compact, it suffices to show that S is compact.

Suppose that S is not compact. Then there is an open cover $\{U_i : i \in I\}$ of which there is no finite subcover of S. Let us divide the square S into four smaller squares by joining the

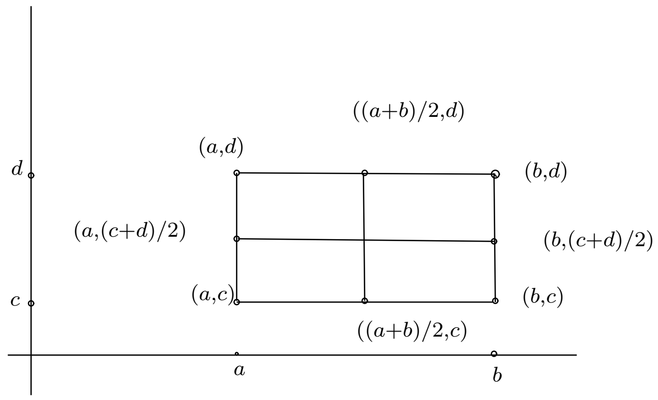

Figure 1:

pairs of midpoints of opposite sides. (See Exercise 14 above.) One of these square will not have a finite subcover from the given cover. For, otherwise, all these four squares will have finite subcovers so that S itself will admit a finite subcover. Choose one such smaller square and call it S_1. Note that the length of its sides is R and that if (a_1, c_1) is the bottom left vertex of S_1, then $a_1 \geq a_0 = -R$ and $c_1 \geq c_0 = -R$. We repeat the argument by subdividing S_1 into four squares and choosing one of the smaller squares which does not admit a finite subcover of $\{U_i\}$. Call this smaller square as S_2. Note that the length of its sides is $R/2$ and that if (a_2, c_2) is the bottom left vertex of S_2, then $a_1 \leq a_2$ and $c_1 \leq c_2$.

Proceeding recursively, we have a sequence of squares S_n such that S_n dose not admit a finite subcover and the length of sides of S_n is $2^{-n+1}R$ and its bottom left vertex (a_n, c_n) is such that $a_{n-1} \leq a_n$ and $c_{n-1} \leq c_n$. Thus we have two sequences of real numbers (a_n) and (c_n). They are bounded and monotone. Hence there exist real numbers a and c such that $a_n \to a$ and $c_n \to c$. It follows that $(a_n, c_n) \to (a, c) \in \mathbb{R}^2$. Since S is closed, we infer that $(a, c) \in S$. Hence there is U_{i_0} in the open cover such that $(a, c) \in U_{i_0}$. Since U_{i_0} is open there exists an $r > 0$ such that $B((a, c), r) \subset U_{i_0}$.

Choose $n \in \mathbb{N}$ so that (1) $\operatorname{diam} S_n = 2^{-n+1}\sqrt{2}R < r/2$ and (2) $d((a, c), (a_n, c_n)) < r/2$. We then have, for any $(x, y) \in S_n$,

$$d((a, c), (x, y)) \leq d((a, c), (a_n, c_n)) + d((a_n, c_n), (x, y)) < r/2 + 2^{-n+1}\sqrt{2}R < r.$$

Thus $S_n \subset B((a, c), r) \subset U_{i_0}$. But then $\{U_{i_0}\}$ is a finite subcover for S_n, contradicting our choice of S_k's. Therefore, our assumption that S is not compact is not tenable. □

Remark 15. The proof carries through for all dimensions n. Let a cube $Q = \prod_{k=1}^{n}[a_i, b_i]$ be given. Let us call the point $(a_1, \dots, a_n)$ as the left bottom corner of the cube. If we bisect each of the sides $[a_i, b_i]$ of this cube, we get 2^n subcubes. The noteworthy feature is that the if $(c_1, \dots, c_n)$ is the left bottom corner of any of these subcubes, then $a_i \leq c_i$ for $1 \leq i \leq n$. The rest of the proof is as above.

Proof of Theorem 2: To prove the first part, consider the sets $U_n := \{x \in X : |f(x)| < n\}$ for $n \in \mathbb{N}$. Clearly, $U_n \subset U_{n+1}$. Also, since f is continuous, and

$$U_n = f^{-1}(B(0, n)), \text{ the inverse image of the open ball },$$

we conclude that U_n is open. Furthermore, by Archimedan property of $\mathbb{R}$, given any $x \in X$, there exists $n \in \mathbb{N}$ such that $|f(x)| < n$. Hence $x \in U_n$. In other words, $\{U_n : n \in \mathbb{N}\}$ is an open cover of the compact set X. By compactness, we can find a finite subcover $U_{n_1}, \ldots, U_{n_k}$ of X. If we let $N = \max\{n_i : 1 \leq i \leq k\}$, it follows (since (U_n) is an increasing sequence of subsets) that $X \subset U_N$. Thus, if $x \in X$, $x \in U_N$ or which is same as saying $|f(x)| < N$ for all $x \in X$. The first part is completely proved.

To prove the second part, let us assume that f is real valued. The subset $\{f(x) : x \in X\} \subset \mathbb{R}$ is a nonempty bounded subset of $\mathbb{R}$ by the first part. Hence $\sup\{f(x) : x \in X\} \in \mathbb{R}$. Call it M. We claim that there exists an $a \in X$ such that $f(a) = M$. Suppose that this is false. This means that for any $x \in X$, we have $f(x) < M$. We consider the following sets:

$$U_n := \{x \in X : f(x) < M - 1/n\}.$$

We note the following facts about U_n. (i) $U_n \subset U_{n+1}$ or more generally, $U_m \subset U_n$ for all $m < n \in \mathbb{N}$. (ii) U_n is open for any n. For, U_n is the inverse image of the open set $(-\infty, M - 1/n) \subset \mathbb{R}$ under the continuous map f. (iii) The family $\{U_n : n \in \mathbb{N}\}$ is an open cover of X. We need only show that $X \subset \cup_n U_n$. Given any $x \in X$, since $f(x) < M$, $M - f(x) > 0$. By the Archimedean property of $\mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n(M - f(x)) > 1$. Hence $M - f(x) > 1/n$ or $f(x) < M - 1/n$. We conclude that $\in U_n$.

Now we proceed as earlier to conclude that $X = U_N$ for some $N \in \mathbb{N}$. (For, by compactness of X, we can extract a finite subcover, say, $U_{n_1}, \ldots, U_{n_k}$ of X. Let $N := \max\{n_1, \ldots, n_k\}$. Using property (i) of U_n's, it follows that $U_{n_j} \subset U_N$ for $1 \leq j \leq k$. This means that

$$X \subset \cup_{j=1}^{k} U_{i_j} = U_N.$$

This leads us to conclude that $f(x) < M - 1/N$ for all $x \in X$. But then l.u.b. $\{f(x) : x \in X\} \leq M - 1/N$, contradicting our assumption that the supremum f on X is M. Thus we are forced to conclude that our assumption that there exists no $x \in X$ such that$f(x) = M$ is incorrect.

One similarly proves that there exists $b \in X$ such that $f(b) \leq f(x)$ for all $x \in X$. $\square$