Compact Spaces

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

Theorem 1. $[a, b]$ *is compact.*

Proof. Given an open cover $\mathcal{U}$ of $[a, b]$, let

$$E := \{x \in [a, b] \mid [a, x] \text{ is covered by finitely many elements of } \mathcal{U}\}.$$

We note that $E \neq \emptyset$, since $a \in E$: For, $[a, a] = \{a\}$ and since $\mathcal{U}$ is an open cover there exists $U \in \mathcal{U}$ such that $a \in U$. Hence $[a, a]$ is covered by the single element U.

E is bounded by b. Hence the supremum of E, say β exists.

We claim that $\beta \in E$ and that $\beta = b$. The claim proves the result. Suppose the claim is false.

Now $\beta \in [a, b]$ since $[a, b]$ is closed. There exists $V \in \mathcal{U}$ such that $\beta \in V$. Hence there exists $\varepsilon > 0$ such that $(\beta - \varepsilon, \beta + \varepsilon) \subseteq V$, as V is open. Assume that $\beta \neq b$. Then we may assume that ε is so small that $(\beta - \varepsilon, \beta + \varepsilon) \subseteq [a, b]$. Since $\beta = \sup E$, $\beta - \varepsilon$ is not an upper bound of E. Thus, there exists $x \in E$, such that $\beta - \varepsilon < x \leq \beta$. Since $x \in E$, there exists finitely many $U_i \in \mathcal{U}$, $1 \leq i \leq n$ such that $[a, x] \subseteq \cup_{i=1}^n U_i$. But then $[a, \beta + \frac{\varepsilon}{2}] \subseteq \cup_{i=1}^n U_i \cup V$. Hence $\beta + \frac{\varepsilon}{2} \in E$, a contradiction since $\beta \geq x$, for all $x \in E$. Hence $\beta = b$. □

Ex. 2. Let $a, b, c, d \in \mathbb{R}$ be such that $b - a = d - c$. Let $S := [a, b] \times [c, d]$ be the square in $\mathbb{R}^2$. The vertices of S are (a, c), (b, c), (b, d) and (a, d). We call the point (a, c) as the bottom left vertex of S. The pair of midpoints of its opposite sides are given by $([a+b]/2, c), ([a+b]/2, d)$ and $(a, [c+d]/2), (b, [c+d]/2])$. By joining the midpoints of opposite sides, we get four smaller squares. Observe that if (a_1, c_1) is the bottom left vertex of any of these squares, we have $a \leq a_1$ and $c \leq c_1$.

Theorem 3. *A subset of* $\mathbb{R}^2$ *is compact iff it is closed and bounded.*

Proof. Let K be a closed and bounded set in $\mathbb{R}^2$. Then there exists $R > 0$ such that $K \subset S := [-R, R] \times [-R, R]$. Since a closed subset of a compact set is compact, it suffices to show that S is compact.

Suppose that S is not compact. Then there is an open cover $\{U_i : i \in I\}$ of which there is no finite subcover of S. Let us divide the square S into four smaller squares by joining the pairs of midpoints of opposite sides. (See the exercise above.) One of these square will not have a finite subcover from the given cover. For, otherwise, all these four squares will have finite subcovers so that S itself will admit a finite subcover. Choose one such smaller square and call it S_1. Note that the length of its sides is R and that if (a_1, b_1) is the bottom left vertex of S_1, then $a_1 \geq a_0 = -R$ and $c_1 \geq c_0 = -R$. We repeat the argument by subdividing S_1 into four squares and choosing one of the smaller squares which does not admit a finite subcover of $\{U_i\}$. Call this smaller square as S_2. Note that the length of its sides is $R/2$ and that if (a_2, c_2) is the bottom left vertex of S_2, then $a_1 \leq a_2$ and $c_1 \leq c_2$.

Proceeding recursively, we have a sequence of squares S_n such that S_n dose not admit a finite subcover and the length of sides of S_n is $2^{-n+1}R$ and its bottom left vertex (a_n, c_n) is such that $a_{n-1} \leq a_n$ and $c_{n-1} \leq c_n$. Thus we have two sequences of real numbers (a_n) and (c_n). They are bounded and monotone. Hence there exist real numbers a and c such that $a_n \to a$ and $b_n \to b$. It follows that $(a_n, c_n) \to (a, c) \in \mathbb{R}^2$. Since S is closed, we infer that $(a, b) \in S$. Hence there is U_{i_0} in the open cover such that $(a, c) \in U_{i_0}$. Since U_{i_0} is open there exists an $r > 0$ such that $B((a, c), r) \subset U_{i_0}$.

Choose $n \in \mathbb{N}$ so that (1) $\operatorname{diam} S_n = 2^{-n+1}\sqrt{2}R < r/2$ and (2) $d((a, c), (a_n, c_n)) < r/2$. We then have, for any $(x, y) \in S_n$,

$$d((a, c), (x, y)) \leq d((a, c), (a_n, c_n)) + d((a_n, c_n), (x, y)) < r/2 + 2^{-n+1}\sqrt{2}R < r.$$

Thus $S_n \subset B((a, c), r) \subset U_{i_0}$. But then $\{U_{i_0}\}$ is a finite subcover for S_n, contradicting our choice of S_k's. Therefore, or assumption that S is not compact is not tenable. □

Theorem 4. *A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.*

Proof. One can adapt the proof of Thm. 3 to prove the theorem including the case when $n = 1$. We leave the details to the reader. □

Theorem 5. *For a metric space (X, d), the following are equivalent:*
(1) X is compact: every open cover has a finite subcover.
(2) X is complete and totally bounded.
(3) Every infinite set has a cluster point.
(4) Every sequence has a convergent subsequence.

Proof. **(1) $\implies$ (2):** Let (X, d) be compact. Given $\varepsilon > 0$, $\{B(x, \varepsilon) \mid x \in X\}$ is an open cover of X. Let $\{B(x_i, \varepsilon) \mid 1 \leq i \leq n\}$ be a finite subcover. Hence X is totally bounded.

Now let (x_n) be a Cauchy sequence in X. Then for every $k \in \mathbb{N}$ there exits n_k such that $d(x_n, x_{n_k}) < \frac{1}{k}$ for all $n > n_k$. Let $U_k := \left\{x \in X \;\middle|\; d(x, x_{n_k}) > \frac{1}{k}\right\}$. Then U_k is open: If $y \in U_k$ and $\delta := d(x_{n_k}, y) - \frac{1}{k}$ then $B(y, \delta) \subseteq U_k$. Now $x_n \notin U_k$ for $n > n_k$. Hence no finite subcover of U_k's cover X: For, if they did, say, $X = \cup_{i=1}^m U_i$ we take $n > \max\{n_1, \ldots, n_m\}$. Then $x_n \notin U_k$ for any k with $1 \leq k \leq m$. This implies that $\{U_k\}$ cannot cover X. Thus there exists $x \in X \setminus \cup_{k=1}^\infty U_k$. But then $d(x, x_{n_k}) < \frac{1}{k}$. Hence $x_{n_k} \to x$. Since (x_n) is Cauchy we see that x_n also converges to $\lim_k x_{n_k}$. Thus X is complete. We have thus shown (1) implies (2).

(2) $\implies$ (3): Let E be an infinite subset of X. Let F_n be a finite subset of X such that $X = \cup_{x \in F_n} B(x, \frac{1}{n})$. Then for $n = 1$ there exists $x_1 \in F_1$ such that $E \cap B(x_1, 1)$ is infinite. Inductively choose $x_n \in F_n$ such that $E \cap (\cap_{k=1}^n B(x_k, \frac{1}{k}))$ is infinite. Since there is $a \in E \cap B(x_m, \frac{1}{m}) \cap B(x_n, \frac{1}{n})$ we see that $d(x_m, x_n) \le d(x_m, a) + d(a, x_n) < \frac{1}{m} + \frac{1}{n} < \frac{2}{m}$ if $m < n$. Thus (x_n) is Cauchy. Since X is complete x_n converges to some $x \in X$. Also $d(x, x_n) < \frac{2}{n}$ for all n. Thus $B(x, 3/n)$ includes $B(x_n, \frac{1}{n})$ which includes infinitely many elements of E. Thus x is a cluster point of E. Hence (3) is proved.

(3) $\implies$ (4): If (x_n) is a sequence in X we let $\{x_n \mid n \in \mathbb{N}\}$ be its image. If this set is finite then (4) trivially follows. So assume that $\{x_n \mid n \in \mathbb{N}\}$ is infinite. Let x be a cluster point of this set. Then there exist elements x_{n_k} such that $d(x, x_{n_k}) < \frac{1}{k}$ for all k. Thus $x_{n_k} \to x$ and (4) is thereby proved.

(4) $\implies$ (1): Let $\{U_\alpha\}$ be an open cover of X. For $x \in X$, let

$$r_x := \sup\{r \in \mathbb{R} \mid B(x, r) \subseteq \text{ for some } U_\alpha\}.$$

We claim that $\varepsilon := \inf\{r_x \mid x \in X\} > 0$. If not there is a sequence (x_n) such that $r_{x_n} \to 0$. But (x_n) has a convergent subsequence, say, $x_{n_k} \to x$. Now $x \in U_\alpha$ for some α and hence there is an $r > 0$ such that $B(x, r) \subset U_\alpha$. For k large enough $d(x, x_{n_k}) < \frac{r}{2}$ so that $r_{x_{n_k}} > \frac{r}{2}$ for all sufficiently large k – a contradiction. Hence the claim is proved.

Let $\varepsilon := \inf\{r_x \mid x \in X\}$. Choose any $x_1 \in X$. Inductively choose x_n such that $x_n \notin \cup_{k=1}^{n-1} B(x_i, \varepsilon/2)$. We cannot do this for all n. For otherwise, (x_n) will not have a convergent subsequence since $d(x_n, x_m) > \frac{\varepsilon}{2}$ for all $m \neq n$. Hence $X = \cup_{k=1}^N B(x_k, \frac{\varepsilon}{2})$ for some N. But then for each k there is an α_k such that $B(x_k, \frac{\varepsilon}{2}) \subset U_{\alpha_k}$. Hence $X = \cup_{k=1}^N U_{\alpha_k}$. Thus $\{U_\alpha\}$ has a finite subcover or X is compact. □

Ex. 6. Prove Thm. 3 using the fourth characterization (in Thm. 5) of compact metric spaces.

Theorem 7 (Tychonoff)**.** *The product of compact spaces is compact. That is, $X := \prod X_\alpha$ is compact if each X_α is compact.*

Proof. Let $\mathcal{F}_0$ be a family of closed sets in X with the finite intersection property (f.i.p). We shall show that there is a point common to all the sets $F \in \mathcal{F}_0$.

We apply Zorn's lemma to get a family $\mathcal{F} \subseteq \mathcal{F}_0$ of (not necessarily closed) sets in X with finite intersection property: Two families $\mathcal{F}$ and $\mathcal{G}$ are related iff $\mathcal{F} \subseteq \mathcal{G}$. Now let $\mathcal{C}$ be any totally ordered chain of families with finite intersection property. That is, if there exists $\mathcal{F}, \mathcal{G} \in \mathcal{C}$, then either $\mathcal{F} \subseteq \mathcal{G}$ or $\mathcal{G} \subseteq \mathcal{F}$. This chain has an upper bound, viz., $\mathcal{H} = \cup_{\mathcal{F} \in \mathcal{C}} \mathcal{F}$, where $\mathcal{H}$ has the finite intersection property. To see that $\mathcal{H}$ has the finite intersection property, let $A_1, \ldots, A_n \in \mathcal{H}$. Then there exists $\mathcal{F}_j \in \mathcal{C}$ such that $A_j \in \mathcal{F}_j \in \mathcal{C}$. Since $\mathcal{C}$ is totally ordered, and $\mathcal{F}_1, \ldots, \mathcal{F}_n$ are finite in number, there exists k with $1 \le k \le n$ such that $\mathcal{F}_j \subseteq \mathcal{F}_k$ for all k. But then $A_1, \ldots, A_n \in \mathcal{F}_k$ and $\mathcal{F}_k$ has the finite intersection property. Hence $A_1 \cap \cdots \cap A_n \neq \emptyset$.

Hence by Zorn's lemma, there exists a maximal family $\mathcal{F} \in \mathcal{C}$, with $\mathcal{F} \supseteq \mathcal{F}_0$. Let $\mathcal{F}^\alpha$ denote $\{E^\alpha := P_\alpha(E), \quad E \in \mathcal{F}\}$. Then $\mathcal{F}_\alpha \subseteq P(X_\alpha)$ has the finite intersection property, (here $P_\alpha \colon X \to X_\alpha$ is the canonical projection map). For otherwise, $E_1^\alpha \cap \cdots \cap E_n^\alpha = \emptyset$ will imply $E_1 \cap \cdots \cap E_n = \emptyset$, where $P_\alpha(E_i) = E_i^\alpha$. Hence, $\overline{\mathcal{F}^\alpha} = \{\overline{E^\alpha}\}$ has finite intersection property.

Since X_α is compact, there exists $x_\alpha \in \cap \overline{E^\alpha}$ where the intersection is over all $E^\alpha \in \mathcal{F}^\alpha$. Let $x \in \prod X_\alpha$ be such that $x(\alpha) := x_\alpha$.

Claim: $x \in \cap_{F \in \mathcal{F}} \overline{F}$.

Since $\mathcal{F} \supseteq \mathcal{F}_0$, the claim completes the proof of the theorem.

Proof of the Claim:

Let U be an open set in X. By definition of product topology, there exists $\alpha_1, \ldots, \alpha_n$ and open sets $U_{\alpha_i} \subseteq X_{\alpha_i}$, $1 \leq i \leq n$ such that $x \in \cap_{i=1}^n P_{\alpha_i}^{-1}(U_{\alpha_i}) \subseteq U$ with $x \in \cap_{i=1}^n P_{\alpha_i}^{-1}(U_{\alpha_i})$. This implies $x_{\alpha_i} \in U_{\alpha_i}$ for all i. By hypothesis on x_α's, $x_{\alpha_i} \in \overline{F}_{\alpha_i}$ for all $F_{\alpha_i} \in \mathcal{F}^{\alpha_i}$. That is, $U_{\alpha_i} \cap \overline{F}_{\alpha_i} \neq \emptyset$, for all $F_{\alpha_i} \in \mathcal{F}^{\alpha_i}$. Hence $P_{\alpha_i}^{-1}(U_{\alpha_i})$ has a non-empty intersection with every $F \in \mathcal{F}$. Therefore $P_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}$ (otherwise $\mathcal{F} \cup \{P_{\alpha_i}^{-1}(U_{\alpha_i})\} \supset \mathcal{F}$ and the former has finite intersection property, contradicting the maximality of $\mathcal{F}$). This being true for all i, and $\mathcal{F}$ has finite intersection property, it follows that $\cap_{i=1}^n P_{\alpha_i}^{-1}(U_{\alpha_i}) \in \mathcal{F}$. Since $\mathcal{F}$ has the finite intersection property, this basic open set and hence U intersects each member of $\mathcal{F}$ non-trivially. Since U was an arbitrary open neighborhood of x, this means that $x \in \overline{F}$, for all $F \in \mathcal{F}$. Hence the claim. $\square$

Ex. 8. Prove Thm. 4 using Tychonoff's theorem.