LUB Property of Ordered Fields

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The aim of this article is to make good students of analysis become aware of the equivalent criteria for an ordered field to have LUB Property. Most often, students are under the impression that the Cauchy completeness of an ordered field is equivalent to the LUB property. This article is aimed at the first year students of M.Sc. as the level of sophistication is higher in the article. We prove below that for an ordered field the LUB property, the Bolzano-Weierstrass property, the Heine-Borel property and the fact that any monotone bounded sequence is convergent are equivalent.

To appreciate the main result, we introduce the reader to the field $\mathbb{R}[[X]]$ of formal power series over $\mathbb{R}$. An element of this field is of the form

$$f(X) = a_n X^n + a_{-n+1}X^{-n+1} + \cdots + a_{-1}X^{-1} + a_0 + a_1 X + a_2 X^2 + \cdots ,$$

where the coefficients $a_k \in \mathbb{R}$, for $k \in \mathbb{Z}$. We can also think of them as functions $a\colon \mathbb{Z} \to \mathbb{R}$ such that there exists $n_0(a) \in \mathbb{N}$ such that $a_{-n} = 0$ if $n > n_0(a)$. Or equivalently, we can think of them as doubly infinite sequence

$$(\ldots, 0, 0, 0, a_{-n}, a_{-n+1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_m, , \ldots).$$

But the first expression is quite important, as it allows us to multiply two such elements. Let $f(X) := \sum_{k\in\mathbb{Z}} a_k X^k$ and $g(X) := \sum_{r\in\mathbb{Z}} b_r X^r$. Then we add and multiply them out as we do in the case of polynomials:

$$\begin{aligned} f(X) + g(X) &:= \sum_m (a_m + b_m) X^m \\ f(X)\cdot g(X) &:= \sum_{n\in\mathbb{Z}} (\sum_{r+s=n} a_r b_s) X^n. \end{aligned}$$

One can show that with these operations, $\mathbb{R}[[X]]$ becomes a field. We introduce an order on this field as follows: We compare the coefficients of the lowest power of X. If they are equal, then we move on to the next higher power and so on. You may also think of this as the lexicographic order on the set of all doubly infinite sequences in F.

In any ordered field F, we can define the absolute value $|x|$ for any $x \in F$ in the usual manner. This absolute value defines a metric on F again in the standard way: $d(x, y) := |x - y|$. We say that a sequence (x_n) in F is convergent if there exists an $x \in F$ such that for every positive $\varepsilon \in F$, there exists an $n_0 \in \mathbb{N}$ such that $|x - x_n| \le \varepsilon$ for all $n \ge n_0$. Cauchy sequences are defined similarly.

Ex. 1. Show that $\mathbb{R}[[X]]$ is complete in the sense of Cauchy: every Cauchy sequence is convergent. *Hint:* Note that a small positive element of this field will have nonzero coefficients for X^k for all $k \leq N$ for very large N. Use this to conclude that any Cauchy sequence will have the property that for any fixed k, there exists $n(k)$ such that the coefficients of X^k in the k-th terms of the sequence, for $k \geq n(k)$, becomes a constant.

Ex. 2. Show that $\mathbb{R}[[X]]$ does not have Archimedean property by showing that $\mathbb{N}$ is bounded above by $f(X) = 1/X$. Hence conclude that $\mathbb{R}[[X]]$ does not enjoy the LUB property.

Theorem 3 (Heine-Borel)**.** *Every open cover of a bounded and closed subset in an ordered field with the LUB property admits a finite subcover.*

Proof. Let E be a closed and bounded subset. Let $\{U_\alpha : \alpha \in A\}$ be an open cover of E. For each element x of the field, let

$$E_x := \{y \in E : y \leq x\}.$$

Now let

$$C := \{x : E_x \text{ is covered by finitely many } U_\alpha\}.$$

Since E is closed and bounded, $b := \inf E \in E$. (For, if $b = \inf E$, then for each $n \in \mathbb{N}$, $b + 1/n$ is not a lower bound for E and hence there exists $y_n \in E$ such that $b \leq y < b + 1/n$. By Sandwich lemma, $y_n \to b$. Since E is closed, $b \in E$.)

Now, $E_b = \{b\}$ and hence $b \in C$. If C has no upper bound, then it contains an $x > \sup E$. For this x, $E_x = E$ and hence the theorem is proved.

Let, if possible, C be bounded above. Let $x_0 := \sup C$. If, for some $\varepsilon > 0$, the interval $(x_0 - \varepsilon, x_0 + \varepsilon) \cap E = \emptyset$, then $E_{x_0 - \varepsilon} = E_{x_0 + \varepsilon} = E_x$ for all $x_0 - \varepsilon < x < x_0 + \varepsilon$. Since x_0 is the least upper bound for C, there exists an $x \in C$ such that $x_0 - \varepsilon < x < x_0 + \varepsilon$. But then E_x and hence $E_{x_0+\varepsilon}$ admits a finite subcover. It follows that $x_0 + \varepsilon \in C$, contradicting the fact that $x_0 = \sup C$. We therefore are led to conclude that for each $\varepsilon > 0$, there exists a point of E in $(x_0 - \varepsilon, x_0 + \varepsilon)$. That is, x_0 lies in the closure of E which is E, since E is closed.

If $x_0 \in E$, then there exists α_0 such that $x_0 \in U_{\alpha_0}$. Since U_{α_0} is open, there exists a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset U_\alpha$. Since $x_0 - \delta$ is not an upper bound for C, there exists $x \in C$ such that $x_0 - \delta < x$. But E_x is covered finitely many U_α's, say, by U_{α_k}, $1 \leq k \leq n$. But then $\{U_{\alpha_r} : 0 \leq r \leq n\}$ is a finite subcover of $E_{x_0+\delta/2}$. Hence $x_0 + \delta/2 \in C$, again a contradiction. This forces us to conclude that C is unbounded. □

Ex. 4. Exhibit a bounded closed subset of $\mathbb{R}[[X]]$ with an open cover which does not admit a finite subcover.

Theorem 5 (Bolzano-Weierstrass)**.** *Every bounded infinite subset of an ordered field in which Heine-Borel theorem is true has a cluster point.*

Proof. Assume that an infinite subset A has no cluster point. Then, for each $x \in A$, there exists $\varepsilon_x > 0$ such that $(x - \varepsilon_x, x + \varepsilon_x)$ has no point of A other than x. Then $\{(x - \varepsilon_x, x + \varepsilon_x) : x \in A\}$ is an open cover of A. It does not admit any finite subcover. Since A has no cluster point, it is automatically closed. Hence we have produced a closed and bounded subset and an open cover of it which does not admit a finite subcover. This is a contradiction to our assumption that the filed enjoys the Heine-Borel property. □

Ex. 6. Show that Bolzano-Weierstrass property fails in $\mathbb{R}[[X]]$.

Lemma 7. *Every bounded monotone sequence in an ordered field in which Bolzano Weierstrass theorem holds is convergent.*

Proof. Assume that (x_n) is nondecreasing. If the range $\{x_n\}$ is finite, let $x := \max\{x_n\}$. Then $x = x_{n_0}$ and hence $x_n = x$ for all $n \geq n_0$, since (x_n) is nondecreasing. In this case, $\lim_n x_n = x$.

If the range is infinite, then there is a cluster point, say, x by the Bolzano-Weierstrass property. We claim that x is an upper bound for the range. For, if not, there exists n_0 such that $x_{n_0} > x$. Since $x_n < x_{n_0}$ only if $n < n_0$, it follows that x is a cluster point of the finite set $\{x_n : 1 \leq n \leq n_0\}$, which is impossible. Hence x is an upper bound for $\{x_n\}$. We now show that $\lim x_n = x$. Given $\varepsilon > 0$, there exists n_0 such that $x_{n_0} \in (x-\varepsilon, x+\varepsilon)$. Now , we have

$$x - \varepsilon < x_{n_0} \leq x_n \leq x < x + \varepsilon \text{ for } n \geq n_0.$$

□

Ex. 8. Give an example of a bounded monotone sequence in $\mathbb{R}[[X]]$ which is not convergent.

Lemma 9. *If F is an ordered field in which every bounded monotone sequence has a limit, then F has Archimedean property.*

Proof. Assume that there exists an $a > 0$ and $b > 0$ in F such that $na \leq b$ for all $n \in \mathbb{N}$. Then the sequence (a_n) where $a_n := na$ is monotone and bounded. Hence by hypothesis, $\lim a_n$ exists in F. Call this limit c. Note that c is the least upper bound for the set $\{a_n\}$. We have

$$c \geq (n+1)a = na + a, \text{ and hence } c - a \geq na = a_n \qquad \forall n \in \mathbb{N}.$$

This shows that $c - a$ is an upper bound for $\{a_n\}$. Since $a > 0$, this is a contradiction to the fact that c is the least upper bound for $\{a_n\}$. □

Lemma 10. *Let F be an ordered field in which every bounded monotone sequence is convergent. Then F has the LUB property.*

Proof. Let A be a nonempty set bounded above, say, by b. Let $a \in A$. If $(a+b)/2$ is an upper bound for A, set

$$a_1 = a, \qquad b_1 = \frac{a+b}{2};$$

If $(a+b)/2$ is not an upper bound, then set

$$a_1 = \frac{a+b}{2}, \qquad b_1 = b.$$

Now set $(a_1 + b_1)/2$ equal to b_2 or a_2 according as $(a_1 + b_1)/2$ is or is not an upper bound for A. We continues this way to define monotone sequences (a_n) and (b_n) with the following properties:

(i) b_n is an upper bound for A for each n.

(ii) a_n is not an upper bound for A for each n.

(iii) $b_n - a_n = 2^{-n}(b-a)$ for each n.

Let $c = \lim b_n$. First we note that c is an upper bound for A. For, if $x > c$ holds for some $x \in A$, then $x > b_n$ for some n. This contradicts (i). Finally, if $y < c$, then by (iii), and Lemma 9, $y < a_n$ for some n. It follows from (ii), y is not an upper bound. Hence c is the least upper bound of A. □

Remark 11. It is so obvious that $(b-a)/2^n \to 0$ that we may not appreciate how the Archimedean property enters the proof of the above lemma. See the exercise (Ex. 12) below.

Ex. 12. Compute the (a_n) and (b_n) as in the proof of Lemma 10 where

$$\begin{aligned} a &= (1/X) + 0 + 0X + 0X^2 + \cdots \\ b &= 1 + 0X + 0X^2 + \cdots \end{aligned}$$

How do these sequences behave? What is the relevance of this to the last remark?

We have thus arrived at the main result of this article.

Theorem 13. *Let F be an ordered field. Then the following four properties are equivalent:*
(a) Every nonempty subset of F which is bounded above has a least upper bound in F.
(b) Every open cover of a bounded closed subset of F admits a finite subcover.
(c) Every bounded infinite subset of F has a cluster point.
(d) Every bounded monotone sequence in F has a limit in F.

Proof. Follows from the earlier results.
$(a) \implies (b)$ is Heine-Borel Theorem (Thm. 3).
$(b) \implies (c)$ is Bolzano Weierstrass Theorem (Thm. 5).
$(c) \implies (d)$ is Lemma 7.
$(d) \implies (a)$ is Lemma 10. □

Project: Formulate the nested interval property for an ordered field. Investigate whether it is equivalent to the LUB property.