Properties Equivalent to the LUB Property of Ordered Fields

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

The aim of this article is to make students of analysis become aware of the equivalent criteria for an ordered field to have LUB Property. We prove below that for an ordered field the LUB property, the Bolzano-Weierstrass property, the Heine-Borel property and the fact that any monotone bounded sequence is convergent are equivalent. The article is aimed at students who have gone through a basic course in real analysis.

In the following, we let $\mathbb{F}$ denote an ordered field. The reader may assume $\mathbb{F} = \mathbb{R}$, if it makes him comfortable. In fact, we encourage the readers to draw pictures on the real line whenever possible. However, in any result to be proved he should assume that $\mathbb{R}$ is an ordered filed with only the properties listed in the result. In the examples and exercises, we let $\mathbb{R}$ denote the filed of real numbers learnt in Real Analysis.

We start reviewing the concepts of least upper bound (l.u.b. or sup) and greatest lower bound (g.l.b. or inf) of bounded sets in $\mathbb{F}$.

Definition 1. A subset $E \subset \mathbb{F}$ is said to be *bounded above* if there exists $\alpha \in \mathbb{F}$ such that $x \leq \alpha$ for all $x \in E$. We then say α is an *upper bound* for E.

A set bounded below and its lower bound are defined analogously.

Ex. 2. If α is an upper bound for E and if $\beta \geq \alpha$, then β is an upper bound for E. What is the analogous statement for lower bounds?

The following definition is motivated by the above exercise.

Definition 3. Let $\emptyset \neq E \subset \mathbb{F}$ be bounded above. We say $\alpha \in \mathbb{F}$ is *the least upper bound* of E if (i) α is an upper bound for E and (ii) if β is an upper bound for E, then $\beta \geq \alpha$.

(ii) can be replaced by (ii)$'$: if $\alpha' < \alpha$, then α' is not an upper bound for E, that is, given $\alpha' < \alpha$, there exists $x \in E$ such that $x > \alpha'$. This formulation is quite useful.

Greatest lower bound of a set bounded below is analogously defined.

Least upper bounds and greatest lower bounds, if they exist, are unique.

Definition 4. We say that an ordered field $\mathbb{F}$ is *order complete* or $\mathbb{F}$ has the LUB property if $E \subset \mathbb{F}$ is a nonempty set bounded above, then there exists a least upper bound for E **in** $\mathbb{F}$.

Ex. 5. Give an example to show that the least upper bound of a set need not be an element of the set.

Ex. 6. If $x \in E$ is an upper bound for E, then x is the least upper bound of E.

Definition 7. A subset $A \subset \mathbb{R}$ is said to be *closed* in $\mathbb{R}$ if for any sequence (x_n) in A such that x_n converges to an $x \in \mathbb{R}$, then $x \in A$.

Example 8. The closed interval $[a, b]$ is closed in $\mathbb{R}$. For, if (x_n) is a sequence in $[a, b]$ and if $x_n \to x \in \mathbb{R}$, then $a \leq x_n \leq b$. Hence $a \leq \lim x_n \leq b$, that is, $x \in [a, b]$.

On the other hand, $(0, 1]$ is not closed. For, consider the sequence $(\frac{1}{n})$. The limit of this sequence is 0 and is not an element of $(0, 1]$.

Ex. 9. Prove that $[a, b)$ is not closed in $\mathbb{R}$.

Ex. 10. Is the set $\mathbb{Q}$ of rational numbers closed in $\mathbb{R}$?

Definition 11. We say that $\mathbb{R}$ enjoys the Archimedean property if the following holds: Let $a > 0$ and x be given in $\mathbb{R}$. Then there exists $n \in \mathbb{N}$ such that $na > x$.

Observe that this is the property which makes us assert that if we fix any length as a unit, we can measure the distance between any two objects in real life!

The Archimedean property is equivalent to saying that the set $\mathbb{N}$ of natural numbers is not bounded above in $\mathbb{R}$. Or equivalently, given any $y \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > y$.

Ex. 12. Using Archimedean property show that $1/n \to 0$ as $n \to \infty$.

Show by induction that $2^n \geq n$ for all $n \in \mathbb{N}$. Hence conclude that $c/2^n \to 0$ as $n \to \infty$ for any fixed $c \in \mathbb{R}$.

Ex. 13. Let $\mathbb{F}$ be order complete. Then it has the Archimedean property.

Observtion 1. Let $\mathbb{F}$ be an ordered field which satisfies the LUB property. Let $E \subset \mathbb{F}$ be closed and bounded. Let $\alpha := \inf E$ and $\beta := \sup E$. We claim that $\alpha, \beta \in E$. For any $n \in \mathbb{N}$, $\beta + \frac{1}{n} > \beta$ and hence is not a lower bound for E. Consequently, there exists $x_n \in E$ such that $x_n < \beta + \frac{1}{n}$. Since β is a lower bound for E, we conclude $\beta \leq x_n < \beta + \frac{1}{n}$. Thus (x_n) is a sequence in E and it is sandwiched between the constant sequence (β) and $(\beta + \frac{1}{n})$. By sandwich lemma, it follows that $x_n \to \beta$. Since E is closed, $\beta \in E$.

We leave the proof of $\alpha \in E$ for the reader.

Note that we needed that $\mathbb{F}$ is Archimedean in the above proof.

Observtion 2. Let $\mathbb{F}$ be order complete. Let $E \subset \mathbb{F}$ be closed. Let $\alpha \in \mathbb{F}$ be such that for any $\varepsilon > 0$, $E \cap (\alpha - \varepsilon, \alpha + \varepsilon) \neq \emptyset$. We then claim $\alpha \in E$. We take $\varepsilon = 1/n$ for any $n \in \mathbb{N}$. Then $(\alpha - 1/n, \alpha + 1/n) \cap E \neq \emptyset$. Pick any element in this intersection and denote it by x_n. Then (x_n) is a sequence in E and $x_n \to \alpha$. Since E is closed, we conclude that $\alpha \in E$.

Definition 14. Let $E \subset \mathbb{F}$. Let $\{J_\alpha : \alpha \in \Lambda\}$ be a collection of open intervals index by Λ. (Λ is called the indexing set of the family.) We say that $\{J_\alpha : \alpha \in \Lambda\}$ is an open cover of E if each J_α is an open interval and $E \subset \cup_{\alpha \in \Lambda} J_\alpha$.

We say that the given open cover of E admits a finite subcover if we can find a finite subset $F \subset \Lambda$ such that $E \subset \cup_{\alpha \in F} J_\alpha$. If we let $F = \{\alpha_1, \dots, \alpha_n\}$, then this can be written as $E \subset J_{\alpha_1} \cup \cdots \cup J_{\alpha_n}$.

Example 15. Let $E = (0, 1]$. Let $J_n := (1/n, 2)$ for $n \in \mathbb{N}$. We claim that $\{J_n : n \in \mathbb{N}\}$ is an open cover of E. For, let $x \in (0, 1]$. We are required to show that $x \in \cup_{n\in\mathbb{N}} J_n$. That is, we have to find an N such that $x \in (1/N, 2)$. If such an N exists, then $1/N < x \leq 1$ so that $N > 1/x$. By Archimedean property of $\mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > 1/x$ so that $1/N < x$. Clearly, $x \in J_N$.

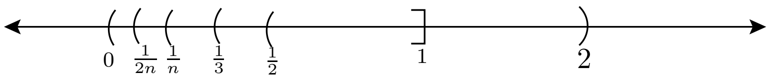

Figure 1: Picture for Example 15

We now ask a question. Does this open cover admit a finite subcover? That is, does there exists finitely many indices, say, $n_1, n_2, \dots, n_k$ such that

$$E \subset J_{n_1} \cup J_{n_2} \cup \cdots \cup J_{n_k}? \tag{1}$$

Note that for any m, we have $J_m \subset J_{m+1}$ and hence $J_m \subset J_n$ if $n \geq m$. In particular, if we let $M := \max\{n_1, \dots, n_k\}$, then

$$J_{n_1} \cup J_{n_2} \cup \cdots \cup J_{n_k} = J_M.$$

Thus if (1) holds, then $(0, 1] \subset (1/M, 2)$. But this is clearly absurd, since, $1/2M \in (0, 1]$, but not in $(1/M, 2)$!

Observe that the set E is bounded but not closed. (Why?) The Heine-Borel theorem proved below says any open cover of a closed and bounded set in an ordered field, which is order complete, admits a finite subcover.

Definition 16. For any set $E \subset \mathbb{F}$, we let

$$E_x := \{y \in E : y \leq x\} = (-\infty, x] \cap E.$$

We refer to this as the initial segment of E determined by x.

Ex. 17. Let $E = \mathbb{Z}$. Given $x \in \mathbb{R}$, what is the initial segment E_x? How do the initial segments $E_0, E_{1/2}, E_{3/4}$ compare?

Observtion 3. Let $E \subset \mathbb{F}$ and $a < b$ be given. Assume that $(a, b) \cap E = \emptyset$. Then for any $s, t \in (a, b)$, we claim that $E_s = E_t$. First, we observe that if $s < t$, then $E_s \subset E_t$. Now, $E_t = E_s \cap (E \cap (s, t])$. Since $E \cap (s, t] \subset E \cap (a, b) = \emptyset$, the claim follows.

Theorem 18 (Heine-Borel)**.** *Let $\mathbb{F}$ be an order complete field. Let E be a closed and bounded subset of $\mathbb{F}$. Let $\{J_\alpha : \alpha \in \Lambda\}$ be a collection of open intervals such that $E \subset \cup_{\alpha\in\Lambda} J_\alpha$. Then we can find finite subset $F \subset \Lambda$ such that $E \subset \cup_{\alpha\in F} J_\alpha$. More concretely, we can find indices $\alpha_1, \dots, \alpha_n \in \Lambda$ such that $E \subset J_{\alpha_1} \cup \cdots \cup J_{\alpha_n}$.*

Proof. Let E be a closed and bounded subset of $\mathbb{F}$. Let $\{J_\alpha : \alpha \in A\}$ be as in the theorem. For each element $x \in \mathbb{F}$, let $E_x := \{y \in E : y \leq x\}$. Now let

$$C := \{x : E_x \text{ is contained in the union of finitely many } J_\alpha\}.$$

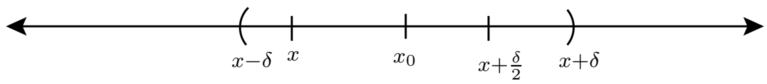

Figure 2: Picture for Theorem 18

Since E is closed and bounded, $b := \inf E \in E$ (by Observation 1).

Now, $E_b = \{b\}$ and hence $b \in C$. Thus C is nonempty. If C has no upper bound, then it contains an $x > \sup E$. For this x, $E_x = E$ and hence the theorem will be proved.

Let, if possible, C be bounded above. Let $x_0 := \sup C$. We shall show that $x_0 \in E$ by showing that for every $\varepsilon > 0$, the set $(x_0 - \varepsilon, x_0 + \varepsilon) \cap E \neq \emptyset$ so that Observation 2 can be invoked. Assume the contrary. If, for some $\varepsilon > 0$, the interval $(x_0 - \varepsilon, x_0 + \varepsilon) \cap E = \emptyset$, then $E_s = E_t = E_x$ for all $s, t, x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ by Observation 3. Since x_0 is the least upper bound for C, there exists an $x \in C$ such that $x_0 - \varepsilon < x < x_0 + \varepsilon$. But then E_x and hence $E_{x_0+\varepsilon/2}$ admits a finite subcover. It follows that $x_0 + \varepsilon/2 \in C$, contradicting the fact that $x_0 = \sup C$. We therefore are led to conclude that for each $\varepsilon > 0$, there exists a point of E in $(x_0 - \varepsilon, x_0 + \varepsilon)$. That is, $x_0 \in E$ by Observation 2.

Since $x_0 \in E$, there exists α_0 such that $x_0 \in J_{\alpha_0}$. Since J_{α_0} is open, there exists a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subset J_\alpha$. Since $x_0 - \delta$ is not an upper bound for C, there exists $x \in C$ such that $x_0 - \delta < x$. But E_x is covered finitely many J_α's, say, by J_{α_k}, $1 \leq k \leq n$. But then $\{J_{\alpha_r} : 0 \leq r \leq n\}$ is a finite subcover of $E_{x_0+\delta/2}$. Hence $x_0 + \delta/2 \in C$, a contradiction. This forces us to conclude that C is unbounded. ☐

Definition 19. Let $\mathbb{F}$ be any ordered field. Let $E \subset \mathbb{F}$. We say a point $x \in \mathbb{F}$ is a cluster point or an accumulation point of E if for each $\varepsilon > 0$, the interval $(x - \varepsilon, x + \varepsilon)$ contains a point of E **other than** x.

Such an x is a super-star or celebrity of the set E thought of as a community. In any small neighbourhood around such a celebrity, we shall always find people from E other than him/herself.

Example 20. The set of natural numbers $\mathbb{N}$ has no cluster point in $\mathbb{R}$. For, if $x \in \mathbb{N}$ is a cluster point of $\mathbb{N}$, then if we choose $\varepsilon = 1$, then $(x - 1, x + 1)$ has only one point common with $\mathbb{N}$, namely x itself. If $x \notin \mathbb{N}$ is a cluster point of $\mathbb{N}$, let n be the greatest integer less than x. Let $\varepsilon := \min\{x - n, n + 1 - x\}$. Then $(x - \varepsilon, x + \varepsilon)$ has no point of $\mathbb{N}$. Thus no point of $\mathbb{R}$ is a cluster point of $\mathbb{N}$.

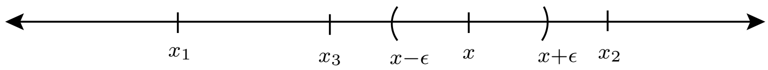

Figure 3: Picture for Obs. 4

Observtion 4. Let $\mathbb{F}$ be any ordered field. Let $E \subset \mathbb{F}$ be a finite set. Then E has no cluster point in $\mathbb{F}$. Let $E = \{x_1, \dots, x_n\}$. For any $1 \leq i \leq n$, x_i is not a cluster point of E. For, if we take

$$\varepsilon < \min\{|x_1 - x_i|, \dots, |x_{i-1} - x_i|, |x_{i+1} - x_i|, \dots, |x_n - x_i|\},$$

then $(x_i - \varepsilon, x_i + \varepsilon) \cap E = \emptyset$. Thus no point of E can be cluster point of E. Similarly, if $x \notin E$ cannot be a cluster point of E. Proof is analogous, we need only take $\varepsilon < \min\{|x - x_k| : 1 \leq k \leq n\}$.

Observtion 5. This is almost similar to the last observation. If x is a cluster point of a set E, then for every $\varepsilon > 0$, $(x - \varepsilon, x + \varepsilon) \cap E$ must be an infinite set. For, if not, let x_k, $1 \leq k \leq n$ be the elements of E (other than x) which lie in the intersection. Then if we define $\delta := \min\{|x_1 - x|, \dots, |x_i - x|, \dots, |x_n - x|\}$, then $(x - \delta, x + \delta) \cap E \subset (x - \varepsilon, x + \varepsilon) \cap E$ and hence is at most $\{x\}$, a contradiction.

Ex. 21. Let $E = \mathbb{Q}$. Then every real number is a cluster point of E.

Ex. 22. Let J be any type of nonempty interval in $\mathbb{R}$. Then each element of J is a cluster point of J. When can you find a cluster point of J outside J?

Observtion 6. Let E be a set such that there is no cluster point of E in $\mathbb{F}$. Then we claim that E is closed. In fact, we shall show in such a situation, if a sequence (x_n) of E converges to x, then the sequence (x_n) is "essentially" the constant sequence (x). For, as x is not a cluster point of E, there exists $\varepsilon > 0$ such that

$$(x - \varepsilon, x + \varepsilon) \cap E \text{ contains at most } x \text{ only.} \tag{2}$$

Now, if (x_n) is a sequence in E such that $x_n \to x$, then by the definition of convergence, for ε as above, we can find $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $x_n \in (x - \varepsilon, x + \varepsilon)$. It follows from (2) that $x_n = x$ for all $n \geq n_0$. That is, $x \in E$.

Theorem 23 (Bolzano-Weierstrass)**.** *Every bounded infinite subset of an ordered field $\mathbb{F}$ in which Heine-Borel theorem is true has a cluster point.*

Proof. Assume that an infinite subset A has no cluster point. Then, for each $x \in A$, there exists $\varepsilon_x > 0$ such that $(x - \varepsilon_x, x + \varepsilon_x)$ has no point of A other than x. Then $\{(x - \varepsilon_x, x + \varepsilon_x) : x \in A\}$ is an open cover of A. It does not admit any finite subcover. Since A has no cluster point, it is closed by Observation 6. Hence we have produced a closed and bounded subset and an open cover of it which does not admit a finite subcover.[Why? Reason: If there is a finite subcover, then we can find $x_1, \dots, x_n$ such that $E \subset J_1 \cup \cdots J_n$ where $J_k := (x_k - \varepsilon_{x_k}, x_k + \varepsilon_{x_k})$, $1 \leq k \leq n$. Hence we have

$$E = E \cap E \subset (E \cap J_1) \cup \cdots \cup (E \cap J_n) = \{x_1, \dots, x_n\},$$

a contradiction, since E is infinite.]

This is a contradiction to our assumption that $\mathbb{R}$ enjoys the Heine-Borel property. ☐

Lemma 24. *Every bounded monotone sequence in an ordered field $\mathbb{F}$ in which Bolzano Weierstrass theorem holds is convergent.*

Proof. Recall that any sequence in $\mathbb{F}$ is a function form $\mathbb{N}$ to $\mathbb{F}$. The range of the function is the set $\{x_n : n \in \mathbb{N}\}$.

Assume that (x_n) is nondecreasing. If the range $\{x_n\}$ is finite, let $x := \max\{x_n\}$. Then $x = x_{n_0}$ and hence $x_n = x$ for all $n \geq n_0$, since (x_n) is nondecreasing. In this case, $\lim_n x_n = x$.

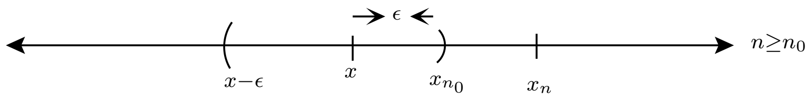

Figure 4: Picture of Lemma 24

If the range is infinite, then there is a cluster point, say, x by the Bolzano-Weierstrass property. We claim that x is an upper bound for the range. For, if not, there exists n_0 such that $x_{n_0} > x$. Since (x_n) is nondecreasing, it follows that $x_n \geq x_{n_0} > x$ for all $n \geq n_0$. If we choose $\varepsilon < x_{n_0} - x$, then $x_n \notin (x_{n_0} - \varepsilon, x_{n_0} + \varepsilon)$ for $n \geq n_0$. Thus it follows that x must be a cluster point of the finite set $\{x_k : 1 \leq k < n_0\}$, which is impossible by Observation 4. Hence x is an upper bound for $\{x_n\}$.

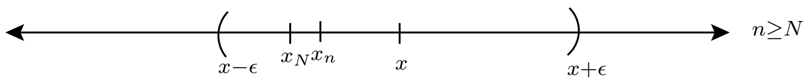

Figure 5: 2nd Picture for Lemma 24

We now show that $\lim x_n = x$. Given $\varepsilon > 0$, since x is a cluster point of the range of the sequence, there exists $N \in \mathbb{N}$ such that $x_N \in (x - \varepsilon, x + \varepsilon)$. Now for $n \geq N$, since the sequence is nondecreasing, we have $x_N \leq x_n$. Since x is an upper bound for $\{x_n\}$, we also have $x_n \leq x < x + \varepsilon$. Putting all these together, we have

$$x - \varepsilon < x_N \leq x_n \leq x < x + \varepsilon \text{ for } n \geq N.$$

Thus, x_n converges to x. □

Ex. 25. Let $\mathbb{F}$ be an ordered field. Assume that (a_n) is nondecreasing sequence. Assume that $a_n \to c$ for $c \in \mathbb{F}$. Show that c is the least upper bound of the set $\{a_n\}$.

Lemma 26. *If $\mathbb{F}$ is an ordered field in which every bounded monotone sequence has a limit, then $\mathbb{F}$ has Archimedean property.*

Proof. Assume that there exists an $a > 0$ and $b > 0$ in $\mathbb{F}$ such that $na \leq b$ for all $n \in \mathbb{N}$. Then the sequence (a_n) where $a_n := na$ is monotone and bounded. Hence by hypothesis, $\lim a_n$ exists in $\mathbb{F}$. Call this limit c. By Ex. 25, c is the least upper bound for the set $\{a_n\}$. We have

$$c \geq (n+1)a = na + a, \text{ and hence } c - a \geq na = a_n \qquad \text{for all } n \in \mathbb{N}.$$

This shows that $c - a$ is an upper bound for $\{a_n\}$. Since $a > 0$, this is a contradiction to the fact that c is the least upper bound for $\{a_n\}$. □

Lemma 27. *Let $\mathbb{F}$ be an ordered field in which every bounded monotone sequence is convergent. Then $\mathbb{F}$ has the LUB property.*

Proof. Let $A \subset \mathbb{F}$ be a nonempty set bounded above, say, by b. Let $a \in A$. If a is an upper bound of A, we stop, for, a is then the least upper bound of A (Ex. 6). So, we assume in the following that $a \in A$ is not an upper bound for A. If $(a+b)/2$ is an upper bound for A, set

$$a_1 = a, \qquad b_1 = \frac{a+b}{2};$$

If $(a+b)/2$ is not an upper bound, then set

$$a_1 = \frac{a+b}{2}, \qquad b_1 = b.$$

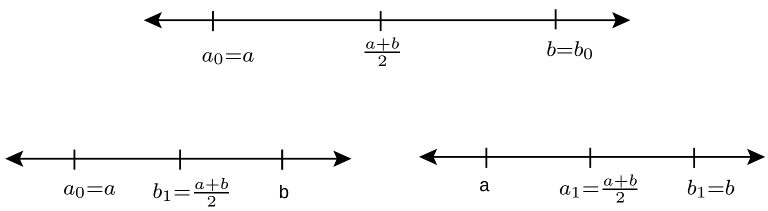

Figure 6: 1st Picture for Lemma 27

Now set $(a_1+b_1)/2$ equal to b_2 or a_2 according as $(a_1+b_1)/2$ is or is not an upper bound for A. We continue this way to define monotone sequences (a_n) and (b_n) with the following properties:

(i) b_n is an upper bound for A for each n.
(ii) a_n is not an upper bound for A for each n.
(iii) $b_n - a_n = 2^{-n}(b-a)$ for each n.

Let $c = \lim b_n$. We claim that c is the least upper bound for A.

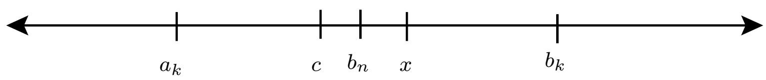

Figure 7: 2nd Picture for Lemma 27

First we note that c is an upper bound for A. For, if $x > c$ holds for some $x \in A$, then $x > b_n$ for some n. [Reason: if $\varepsilon := x - c$, since $b_n \to c$, there exists $n \in \mathbb{N}$ such that $b_n \in (c-\varepsilon, c+\varepsilon)$, that is, $b_n < c + \varepsilon = x$.] This contradicts (i) and hence we deduce that c is an upper bound of A.

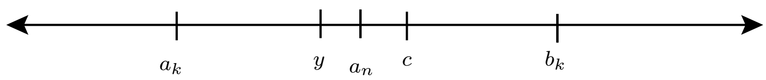

Figure 8: 3rd Picture for Lemma 27

Finally, if $y < c$. We show that y is not an upper bound of A. For, by (iii), and Lemma 26, $y < a_n$ for some n. [Reason: Let $\varepsilon := c - y$. Then there exists m_0 such that $b_n \in (c-\varepsilon, c+\varepsilon)$ for $m \geq m_0$. In particular, $b_m > c - \varepsilon$ for $m \geq m_0$. We now choose $m \geq m_0$ such that

$b_m - a_m = 2^{-m}(b-a) < \varepsilon/2$. (This is possible by the Archimedean property, see Ex. 12.) We observe, for such an m, the following holds:

$$a_m = b_m + (a_m - b_m) > b_m - \varepsilon/2 > b_m - \varepsilon = y.$$

It follows from that y is not an upper bound of A.]

Hence c is the least upper bound of A. □

Remark 28. It is "so obvious" that $(b-a)/2^n \to 0$ that we may not appreciate how the Archimedean property enters the proof of the above lemma.

We have thus arrived at the main result of this article.

Theorem 29. *Let $\mathbb{F}$ be an ordered field. Then the following four properties are equivalent:*
(a) *Every nonempty subset of $\mathbb{F}$ which is bounded above has a least upper bound in F.*
(b) *Every open cover of a bounded closed subset of $\mathbb{F}$ admits a finite subcover.*
(c) *For every bounded infinite subset of $\mathbb{F}$, there exists a cluster point of E in $\mathbb{F}$.*
(d) *Every bounded monotone sequence in $\mathbb{F}$ has a limit in $\mathbb{F}$.*

Proof. Follows from the earlier results.

(a) $\implies$ (b) is Heine-Borel Theorem (Thm. 18).

(b) $\implies$ (c) is Bolzano-Weierstrass Theorem (Thm. 23).

(c) $\implies$ (d) is Lemma 24.

(d) $\implies$ (a) is Lemma 27. □

Remark 30. Most often, students and quite a few experts are under the wrong impression that the Cauchy completeness of an ordered field is equivalent to the LUB property. The rest of the article is devoted to explaining the concepts involved and to showing that an ordered field which is Cauchy complete and **further enjoys** Archimedean property is order complete, that is, it has the LUB property.

Definition 31. Let $\mathbb{F}$ be an ordered field. A sequence (x_n) in $\mathbb{F}$ is said to be Cauchy if for any given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ for $m, n \geq n_0$.

It is easy to show that if (x_n) is a convergent sequence, then it is Cauchy.

If every Cauchy sequence in $\mathbb{F}$ is convergent, then we say that $\mathbb{F}$ is Cauchy complete.

Ex. 32. Let (x_n) be Cauchy in $\mathbb{F}$. Then (x_n) is bounded.

Ex. 33. Let (x_n) be Cauchy. Assume that it has a convergent subsequence (x_{n_k}). Assume that $\lim_{k\to\infty} x_{n_k} = x$ for some $x \in \mathbb{F}$. Show that (x_n) converges to x.

Remark 34. If an ordered field $\mathbb{F}$ has LUB property, then the following results are proved in nay basic course in Real Analysis:

(i) Archimedean property holds in $\mathbb{F}$
(ii) $\mathbb{F}$ is Cauchy complete.

For an elegant simple proof of this can be seen in my article [1] listed below.

Conversely, if the field enjoys the Archimedean property and is Cauchy complete, then it enjoys LUB property. The proof is exactly the same as in (d) $\implies$ (a) of Theorem 29.

Lemma 35. *Let $\mathbb{F}$ be any ordered field. Let (x_n) be any sequence in $\mathbb{F}$. Then it has a monotone subsequence.*

Proof. Consider the set S defined as follows:

$$S := \{n \in \mathbb{N} : x_m < x_n \text{ for } m > n\}.$$

There are two cases: S is finite or infinite.

Case 1. S is finite. Let N be any natural number such that $k \leq N$ for all $k \in S$. Let $n_1 > N$. Then $n_1 \notin S$. Hence there exists $n_2 > n_1$ such that $x_{n_2} \geq x_{n_1}$. Since $n_2 > n_1 > N$, $n_2 \notin S$. Hence we can find an $n_3 > n_2$ such that $x_{n_3} \geq x_{n_2}$. This we way, we can find a monotone nondecreasing (increasing) subsequence, (x_{n_k}).

Case 2. S is infinite. Let n_1 be the least element of S. Let n_2 be the least element of $S \setminus \{n_1\}$ and so on. We thus have a listing of S:

$$n_1 < n_2 < n_3 < \cdots$$

Since n_{k-1} is an element of S and since $n_{k-1} < n_k$, we see that $x_{n_k} < x_{n_{k-1}}$, for all k. We now have a monotone decreasing sequence. □

Proposition 36. *Let $\mathbb{F}$ be an ordered field. Let $\mathbb{F}$ be such that every bounded monotone sequence in $\mathbb{F}$ has a limit in $\mathbb{F}$. Then $\mathbb{F}$ has Archimedean property and is Cauchy complete.*

Proof. That $\mathbb{F}$ has Archimedean property is Lemma 26. Let (x_n) be Cauchy in $\mathbb{F}$. Then (x_n) is a bounded sequence by Ex. 32. Let (x_{n_k}) be a monotone subsequence, which exists by Lemma 35. By hypothesis, $x_{n_k} \to x$ for some $x \in \mathbb{F}$. The result follows from Exercise 33. □

We may therefore modify Theorem 29 as follows

Theorem 37. *Let $\mathbb{F}$ be an ordered field. Then the following five properties are equivalent:*

(a) *Every nonempty subset of $\mathbb{F}$ which is bounded above has a least upper bound in $\mathbb{F}$.*
(b) *Every open cover of a bounded closed subset of $\mathbb{F}$ admits a finite subcover.*
(c) *Every bounded infinite subset of $\mathbb{F}$ has a cluster point in $\mathbb{F}$.*
(d) *Every bounded monotone sequence in $\mathbb{F}$ has a limit in $\mathbb{F}$.*
(e) *The field $\mathbb{F}$ has Archimedean property and is Cauchy complete.*

Proof. (a) $\implies$ (b) $\implies$ (c) $\implies$ (d) is already seen in Theorem 29.

(d) $\implies$ (e) is Prop. 36.

The proof of (e) $\implies$ (a) is exactly the same as the proof of (d) $\implies$ (a) **in** Theorem 29. We outline the arguments. (Confident readers should ignore the rest of the proof and do it on their own!) Let A be a nonempty subset of $\mathbb{R}$ bounded above by b. As earlier, we choose $a \in A$. If a is an upper bound for A, then a is the least upper bound of A. Otherwise, we have two sequences (a_n) and (b_n) with the properties listed in the proof of Theorem 29. It is easily seen that (b_n) is Cauchy. Let c be the limit of (b_n). Then it is an easy exercise to show that c is the least upper bound for A, as earlier. □

Project: Formulate the nested interval property for an ordered field. Investigate whether it is equivalent to the LUB property.

As further reading I suggest the following articles of mine.

[1] The role of LUB Axiom in Real Analysis
[2] LUB Property of Ordered Fields.

It is my sincere hope that these articles will make the readers develop confidence in Real Analysis and prepare them for higher aspects of analysis and topology.

Acknowledgement: This article is based on my article [2] quoted above. I have given more details in this than found in [2] so that this will be accessible to B.Sc. students. I have also removed from [2] an example of an ordered field which serves as a counterexample in many an instance, as the example requires more background and maturity than found in an average B.Sc. student.

This is a very close transcript of my lectures given at Ruia College, Mumbai in July 2002. I thank Professor Mangala Deshpande and the enthusiastic audience.