A Diagnostic Quiz in Complex Analysis

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Notes:
(1) U stands for a connected open subset of $\mathbb{C}$.
(2) Analytic $\equiv$ holomorphic.
(3) References to concepts or theorems, given in brackets, are meant as hints.
(4) $B(a,r) := \{z \in \mathbb{C} : |z-a| < r\}$.

1. Find the real and imaginary part of the functions: (i) $f(z) = \overline{z}$, (ii) $f(z) = e^{-z^2}$.
2. True or false? The function $z \mapsto \overline{z}$ is analytic on $\mathbb{C}$. (Cauchy Riemann equations)
3. What are all the real valued analytic functions on a connect open subset of $\mathbb{C}$? (Cauchy-Riemann equations/Open mapping theorem)
4. Define $\int_\gamma f$ where γ is a continuously differentiable path and $f\colon \mathbb{C} \to \mathbb{C}$ is continuous.
5. Let $f\colon U \to \mathbb{C}$ admit a primitive, say, g on U and γ be a piecewise smooth path in U. What is $\int_\gamma f$?
6. True or false? $|\int_\gamma f| \le \int_\gamma |f|$.
7. Define the radius of convergence of a power series.
8. Let the power series $\sum_{n=0}^\infty c_n z^n$ have $R > 0$ as its radius of convergence. True or false? The sequence of partial sums converge uniformly on the disk $B(0,R)$ of convergence.
9. What is the sum of the series $\sum_{n=1}^\infty n z^{n-1}$? (Term-wise operations on a power series)
10. Let the power series $\sum_{n=0}^\infty c_n z^n$ have $R > 0$ as its radius of convergence. Does it have a primitive in $B(0,R)$, that is, does there exist an analytic function g on $B(0,R)$ such that $f = g'$ on $B(0,R)$? (Term-wise operations on a power series)
11. State a version of Cauchy's theorem you have learnt.
12. Let f be analytic on $B(0,1)$. True or false? There exists an analytic $g\colon B(0,1) \to \mathbb{C}$ such that $g' = f$ on the unit disk. (Cauchy's theorem)
13. Give a primitive of $f(z) = 1/z$ on the upper half-plane $\operatorname{Im} z > 0$. (Branches of logarithm)

14. Let $U := B(i, 1)$. Let γ be any closed path/contour in U. What is the value of $\int_\gamma \frac{dz}{z}$? (Winding number/Cauchy's theorem/Residue theorem)

15. Evaluate (i) $\int_\gamma \frac{z^2+1}{z+1}$ and (ii) $\int_\gamma \frac{e^z}{(z-1)^2}$ where $\gamma(t) = 2e^{it}$. (Cauchy integral formula)

16. Let U be a connected open subset of $\mathbb{C}$ and let f be analytic on U. True or false? The Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$ converges on an open disk $B(a, r_a) \subset U$ for each $a \in U$.

17. Let U be a connected open subset of $\mathbb{C}$ and let f be analytic on U. True or false? f is a power series function on U, that is, there exist $a \in U$ and constants $c_n \in \mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$.

18. True or false? If f is an entire function then there exist constants $c_n \in \mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for all $z \in \mathbb{C}$.

19. Give a complete list of all analytic functions $f\colon \mathbb{C} \to B(0, 1)$. (Liouville's theorem)

20. True or false? Let $f\colon \mathbb{C} \to \mathbb{C}$ be a nonconstant entire function. Then $|f(z)| \to \infty$ as $|z| \to \infty$. (A correct understanding of Liouville's theorem)

21. Any nonconstant polynomial maps $\mathbb{C}$ onto $\mathbb{C}$. (Fundamental theorem of algebra)

22. Let f be a function on U such that for every $\varepsilon > 0$, there exists g analytic on U such that $|f(z) - g(z)| < \varepsilon$ for all $z \in U$. Then f is analytic on U. (Uniform convergence, Morera and Weierstrass theorems)

23. Let $f\colon B(0, 1) \to \mathbb{C}$ be analytic and be such that $f(x) = x$ for all $x \in (0, 1)$. What can you conclude about f? (Identity/Uniqueness theorem)

24. Let $f, g\colon U \to \mathbb{C}$ be analytic. Assume that $f(z) = g(z)$ for all $z \in K$, an infinite compact subset of U. True or false: $f \equiv g$ on U. (Bolzano-Weierstrass and Identity/Uniqueness theorems)

25. Let f be analytic on a connected open set U. If f vanishes on an uncountable subset of U, then f is identically zero. (Bolzano-Weierstrass and identity/uniqueness theorems)

26. Let $f\colon U \to \{z \in \mathbb{C} : |z| \leq 1\}$ be an analytic function which is onto. What can you conclude about f? (Maximum modulus principle)

27. What kind of singularities does $\frac{\sin z}{z}$ have at $z = 0$?

28. What kind of singularities does a nonconstant polynomial have in the extended complex plane? (Laurent expansion at ∞)

29. Which entire functions have essential singularity at ∞? Does an analogue of Casorati-Weierstrass theorem hold for such functions?

30. True or false? If f is defined on $U := \{z \in \mathbb{C} : 0 < |z| < 1\}$ and f has an essential singularity at 0, then $f(U)$ is an open dense set in $\mathbb{C}$. (Casorati-Weierstrass and open mapping theorems).

31. What is the residue at $z = 0$ of an entire function f? of the function $g(z) := f(z)/z$?

32. True or false? Residue theorem is a generalization of the Cauchy integral formula.

33. Let $\gamma(t) = 2e^{it}$, $0 \le t \le 2\pi$. Evaluate $\int_\gamma \frac{e^z}{z-1} dz$. (Residue theorem)

34. Let $P(z)$ be a nonconstant polynomial and $\gamma_R(t) := Re^{it}$, $0 \le t \le 2\pi$. What is the value of $\frac{1}{2\pi i} \int_{\gamma_R} \frac{P'(z)}{P(z)} dz$ for all sufficiently large R? (Argument principle)

35. Find all analytic functions on U that take vales in the hyperbola $\{z \in \mathbb{C} : \mathrm{Re}\, z \cdot \mathrm{Im}\, z = 1\}$. (Open mapping theorem)

36. Let $f\colon U \to \mathbb{C}$ be holomorphic such that its real part is also holomorphic. What can you conclude about f? (Open mapping theorem, twice!)

37. Let $f\colon \mathbb{C}^* \to \mathbb{C}$ be analytic and bounded on $\mathbb{C}^*$. True or false: f is a constant. (Riemann's theorem on removable singularity and open mapping theorem)

38. Let P be a polynomial of degree 2 or greater. Show that $\int_{\gamma_R} \frac{dz}{P(z)} = 0$ for all sufficiently large R. (Residues at ∞)

39. Let $f\colon \mathbb{C} \to \mathbb{C}$ be analytic taking values in the upper half-plane $\{z \in \mathbb{C} : \mathrm{Im}\,(z) > 0\}$. True or false: f is a constant. (Conformal equivalence and Liouville)

40. * Do you understand the following statement? A continuous argument exists on an open connected set U iff a continuous branch of logarithm exists on U.

41. * Do you know the relation between the following concepts? Existence of a continuous branch of logarithm on U and the concept of evenly covered neighbourhoods in the context of covering spaces?