Complex Analysis: Handout-1 (Basic Calculus on $\mathbb{C}$)

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

1 Limits

We shall be brief in this section.

Definition 1. Let $A \subset X$ be a subset of a metric space. Let $x_0 \in X$ be a cluster point of A, not necessarily in A. Let $f: A \to \mathbb{C}$ be a function. We say that $\lim_{x \to x_0} f(x)$ exists if there exists $\ell \in \mathbb{C}$ such that for any given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$x \in A \text{ and } 0 < d(x, x_0) < \delta \Rightarrow |f(x) - \ell| < \varepsilon.$$

It is easy to show that ℓ with this property, if exists, is unique. In such a case, we write $\lim_{x \to x_0} f(x) = \ell$. Note that f need not be defined at x_0.

The study of this concept may be reduced to that of sequences and their limits because of the following exercise.

Ex. 2. With the above notation, $\lim_{x \to x_0} f(x) = \ell$ iff for every sequence (x_n) in A (with $x_n \neq x_0$ for $n \in \mathbb{N}$) such that $\lim x_n = x_0$ we have $\lim f(x_n) = \ell$. *Hint:* Go through the proof of the equivalence of the definitions of continuity in terms of sequences and $\varepsilon - \delta$. (Refer to Theorem **??**.)

Proposition 3. *Let* $\lim_{x \to x_0} f(x) = \alpha$ *and* $\lim_{x \to x_0} g(x) = \beta$. *Then*
(i) $\lim_{x \to x_0} (af + bg)(x) = a\alpha + b\beta$ *for* $a, b \in \mathbb{C}$.
(ii) $\lim_{x \to x_0} (fg)(x) = \alpha\beta$.
(iii) $\lim_{x \to x_0} (f/g)(x) = \alpha/\beta$ *if* $\beta \neq 0$.

Proof. Use the above exercise Ex. 2 and the algebra of limits theorem. □

Ex. 4. Show that $\lim_{z \to i} \frac{z^4 - 1}{z - i} = -4i$.

Ex. 5. Let $f: U \subset \mathbb{C} \to \mathbb{C}$ be given. Show that f is continuous at $z_0 \in U$ iff $\lim_{z \to z_0} f(z) = f(z_0)$.

Definition 6. Let $f : \mathbb{C} \to \mathbb{C}$ be a function. We use the notation $\lim_{z\to\infty} f(z) = \ell$ to mean that if given $\varepsilon > 0$ there exists an $R > 0$ such that for all z with $|z| > R$, we have $|f(z)-\ell| < \varepsilon$.

We use the notation $\lim_{z\to z_0} |f(z)| = \infty$ to say that if for a given $M > 0$, there exists $r > 0$ such that $|f(z)| > M$ for all $z \in B(z_0, r)$.

Note that we have not yet introduced the point at infinity.

Ex. 7. Prove that $\lim_{z\to\infty}(1/z^n) = 0$ for any $n \in \mathbb{N}$.

Ex. 8. Prove or disprove that $\lim_{z\to\infty} |e^{-z}| = 0$.

Ex. 9. Let $p(z) := \sum_{k=0}^n a_k z^k$ be a nonconstant polynomial. Show that there exists $R > 1$ such that $|p(z)| > \frac{1}{2}|a_n||z|^n$ for $|z| \geq R$. In particular, $|p(z)| \to \infty$ as $|z| \to \infty$. *Hint:* Observe that

$$|\frac{p(z)}{z^n}| \geq |a_n| - \sum_{k=0}^{n-1} \frac{|a_k|}{|z|}, \qquad \text{for } |z| \geq 1.$$

2 Functions from $\mathbb{R}$ to $\mathbb{C}$

We deal with differentiation of functions from $\mathbb{R}$ to $\mathbb{C}$.

Definition 10. Let $f : [a, b] \to \mathbb{C}$ be given. We say that f is *differentiable* function at $t \in [a, b]$ if there exists a complex number α such that for a given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$0 \neq h \in \mathbb{R},\ |h| < \delta \ \&\ t + h \in [a, b] \Rightarrow |\frac{f(t + h) - f(t)}{h} - \alpha| < \varepsilon.$$

That is, f is differentiable at t iff $\lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$ exists. If the limit exists and if we denote it by α, then α is unique and called the derivative of f at t. It is denoted by $f'(t)$.

Ex. 11. With the above notation, write $f := u + iv$ where $u(t) := \mathrm{Re}\,(f(t))$ and $v(t) := \mathrm{Im}\,(f(t))$. Then f is differentiable at $t \in [a, b]$ iff the real valued functions $u, v : [a, b] \to \mathbb{R}$ are differentiable at t. Furthermore, $f'(t) = u'(t) + iv'(t)$. Also, if f is differentiable at t, then it is continuous at t.

Remark 12. If f is as above, we may think of f as a differentiable curve in the complex plane: $f(t) = u(t) + iv(t)$ which corresponds to $(u(t), v(t)) \in \mathbb{R}^2$. And $f'(t) \equiv (u'(t), v'(t))$ is thought of as the tangent vector to the curve at t. In physical terms, we may think of $f(t)$ as the position of a particle at time t and $f'(t)$ as the "velocity".

Example 13. Let $z, w \in \mathbb{C}$. Let $\gamma : [0, 1] \to \mathbb{C}$ be given by $\gamma(t) := z + t(w - z)$. Then γ is the line segment $[z, w]$ joining z and w and $\gamma'(t) = w - z$, since $\frac{\gamma(t+h)-\gamma(t)}{h} - (w - z) = 0$, for $h \neq 0$.

Example 14. Let $a \in \mathbb{C}$, $R > 0$. Consider the map $\sigma : [0, 2\pi] \to \mathbb{C}$ given by $\sigma(t) := a + R(\cos t + i \sin t)$. Then σ is the circle with centre at a and radius R. Also, $\sigma'(t) = R(-\sin t + i \cos t)$.

Remark 15. The analogue of mean value theorem is false for complex valued functions of a real variable. For, consider $f : [0, 2\pi] \to \mathbb{C}$ given by $f(t) := \cos t+i \sin t$. Then $f(2\pi)-f(0) = 0$ while $|f'(t)| = 1$ for all $t \in [0, 2\pi]$.

Ex. 16. Let $h : [a, b] \to \mathbb{C}$ be differentiable with $h'(t) = 0$. Then h is a constant. *Hint:* Invoke the result from real analysis to the real and imaginary parts of h.

3 Differentiable Functions on $\mathbb{C}$

Definition 17. Let $U \subset \mathbb{C}$ be open. A function $f\colon U \to \mathbb{C}$ is said to be *differentiable* at $z \in U$ if there exists an $\alpha \in \mathbb{C}$ such that given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\frac{f(z+h) - f(z)}{h} - \alpha\right| < \varepsilon, \quad \text{for } 0 < |h| < \delta.$$

The number α is unique (Exercise!) and called the *derivative* of f at z. It is denoted by $f'(z)$.

Note that f is differentiable at $z \in U$ iff the limit

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists.

If f is differentiable at each $z \in U$, then f is said to be *holomorphic* on U. Let $H(U)$ denote the set of functions holomorphic on U.

If $f \in H(\mathbb{C})$, then f is called an *entire* function.

If $f\colon U \to \mathbb{C}$ is differentiable and if $g := f'\colon U \to \mathbb{C}$ is differentiable at $z \in U$, we denote $f''(z) = g'(z)$. f'' is called the second derivative of f at z. More generally, we define inductively the n-th derivative of f at z by setting $f^{(0)}(z) := f(z)$ and $f^{(n)}(z) := (f^{(n-1)})'(z)$, the derivative of $f^{(n-1)}$ at z. In general, the first, second and third derivatives are denoted by f', f'' and f''' respectively. If $f^{(n)}(z)$ exists for all $n \in \mathbb{N}$ and for all $z \in U$, we say that f is infinitely differentiable on U.

Proposition 18. *Let $f\colon U \to \mathbb{C}$ be given and $z \in U$. Then f is differentiable at z iff there exists a function $f_1\colon U \to \mathbb{C}$, continuous at z and such that $f(w) = f(z) + (w - z)f_1(w)$. Furthermore, $f'(z) = f_1(z)$.*

Proof. Let f be differentiable at z. Define $f_1(w) := \frac{f(w)-f(z)}{w-z}$ for $w \neq z$ and $f_1(z) = f'(z)$. To check the continuity of f_1 at z, let $\varepsilon > 0$ be given. Since f is differentiable at z, for this ε there exists a δ such that if $0 < |w - z| < \delta$, then $|\frac{f(w)-f(z)}{w-z} - f'(z)| < \varepsilon$, that is, $|f_1(w) - f_1(z)| < \varepsilon$ for $|w - z| < \delta$. Hence f_1 is continuous at z.

Conversely, if f_1 exists as specified, observe that $f_1(w)$ must equal $\frac{f(w)-f(z)}{w-z}$ for $w \neq z$. We claim that f is differentiable at z and $f'(z) = f_1(z)$: By continuity of f_1 at z, given $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |w - z| < \delta$, then $|\frac{f(w)-f(z)}{w-z} - f_1(z)| < \varepsilon$. This precisely means that f is differentiable at z and $f'(z) = f_1(z)$. $\square$

Corollary 19. *If f is differentiable at z, then f is continuous at z.*

Proof. Using the notation of the last proposition, we have $f(w) = f(z)+(w-z)f_1(w)$. Hence

$$\lim_{w\to z} f(w) = \lim_{w\to z} (f(z) + (w - z)f_1(w)) = f(z),$$

since f_1 is continuous at z, the second term goes to 0.

Reason: Note that if f_1 is continuous at z, then it is bounded in an open disk around z, say, by M. (See Ex. **??**.) Then $\lim_{w\to z}(w-z)f_1(w) = 0$.

$\square$

Example 20. Let $f\colon U \to \mathbb{C}$ be a constant. Then f is holomorphic on U and $f'(z) = 0$ for all $z \in U$.

4 Integration of functions from $\mathbb{R}$ to $\mathbb{C}$

Let $I = [a, b] \subseteq \mathbb{R}$ and $f\colon I \to \mathbb{C}$ be continuous. Then we define the integral of f by

$$\int_a^b f \equiv \int_a^b f(t)\,dt := \int_a^b \operatorname{Re} f + i \int_a^b \operatorname{Im} f.$$

Here the integrals are Riemann and they exist since $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ are continuous.

Ex. 21. Let $f\colon [0, 2\pi] \to \mathbb{C}$ be given by $f(t) = e^{int}$ for some $n \in \mathbb{Z}$. Compute $\int_0^{2\pi} f(t)\,dt$.

Ex. 22. Let $C([a, b], \mathbb{C})$ denote the complex vector space of continuous functions on $[a, b]$. Show that the map $f \mapsto \int_a^b f(t)\,dt$ is a complex linear map from $C([a, b], \mathbb{C})$ to $\mathbb{C}$.

Theorem 23 (Fundamental Theorem of Calculus)**.**
(i) *Let $f\colon [a, b] \to \mathbb{C}$ be continuous. Define $F(x) := \int_a^x f(t)\,dt$. Then F is differentiable and $F'(x) = f(x)$ for $x \in [a, b]$.*
(ii) *Let $G\colon [a, b] \to \mathbb{C}$ be differentiable with $g := G'$ continuous. Then $\int_a^b g(t)\,dt = G(b) - G(a)$.*

Proof. The strategy is to make use of our definitions of derivative and the integral of a complex valued function of a real variable and make use the fundamental theorem of calculus for real valued functions of a real variable.

Let us write $f(t) = u(t) + iv(t)$. Then u and v are continuous and

$$F(x) = \int_a^x f(t)\,dt = \int_a^x u(t)\,dt + i \int_a^x v(t)\,dt.$$

Now F is differentiable iff $\int_a^x u(t)\,dt$ and $\int_a^x v(t)\,dt$ are so. By the fundamental theorem of calculus, it follows that these two indefinite integrals are differentiable and their derivatives are $u(x)$ and $v(x)$ respectively. Hence $F'(x)$ exists and is equal to $u(x) + iv(x) = f(x)$. This proves (i).

Proof of (ii) is similar and the reader should do it on his own.

Let $H(x) := \int_a^x g(t)\,dt$. Then, $H'(x) = g(x)$ by (i). so that $(H - G)' = 0$. Thus $H - G$ is a constant on $[a, b]$ whence $H(b) - G(b) = H(a) - G(a)$, or $H(b) - H(a) = G(b) - G(a)$. Since $H(a) = 0$, (ii) follows.

$\square$

Ex. 24. Do Ex. 21 now!

Ex. 25. Do Ex. 16. *Hint:* $h(x) - h(a) = \int_a^x h'(t)\,dt$.

Ex. 26. Let $\lambda = a + ib \in \mathbb{C}^*$. Evaluate $\int_0^t e^{\lambda s}\,ds$. Equating the real parts, show that

$$(a^2 + b^2)\int_0^t e^{as}\cos bs\,ds = e^{at}[a\cos bt + b\sin bt] - a.$$

Ex. 27. Let $\gamma\colon [a,b] \to \mathbb{C}$ be a continuously differentiable map with $\gamma(t) \neq z_0$ for all $t \in [a,b]$. (We think of γ as a path not passing through z_0.) Let $g(t) := \int_a^t \frac{\gamma'(s)}{\gamma(s)-z_0}\,ds$, for $t \in [a,b]$. Show that $h(t) := e^{-g(t)}[\gamma(t) - z_0]$ is a constant and hence deduce that

$$\exp(g(t)) = \frac{\gamma(t) - z_0}{\gamma(a) - z_0}, \qquad \text{for } t \in [a,b].$$

What can you say about $g(b)$ if $\gamma(b) = \gamma(a)$? Take $z_0 = 0$ and do you see the meaning of g? We shall return to this exercise later. See Definition **??**, Example **??** and Theorem **??**.

Corollary 28 (Integration by parts)**.** *Let $f, g\colon [a,b] \to \mathbb{C}$ be continuously differentiable. Then*

$$\int_a^b f(t)g'(t)\,dt = [f(t)g(t)]_a^b - \int_a^b f'(t)g(t)\,dt.$$

Proof. This follows from Theorem 23. Recall that $(fg)'(t) = f'(t)g(t) + f(t)g'(t)$. Both sides are continuous functions and so we can integrate them to obtain

$$\begin{aligned}\int_a^b f'(t)g(t)\,dt + \int_a^b f(t)g'(t)\,dt &= \int_a^b (fg)'(t)\,dt \\ &= [f(t)g(t)]_a^b,\end{aligned}$$

by the fundamental theorem of calculus. The result follows from this. □

Proposition 29 (Change of Variable)**.** *Let $h\colon [c,d] \to \mathbb{R}$ be continuously differentiable and $f\colon [a,b] \to \mathbb{R}$ be continuous. Assume that $h([c,d]) \subset [a,b]$. Then*

$$\int_c^d (f \circ h)(s)h'(s)\,ds = \int_{h(c)}^{h(d)} f(t)\,dt.$$

Proof. Consider $\varphi(s) := \int_{h(c)}^{h(s)} f(t)\,dt$. Then φ is the composition of the functions $s \mapsto h(s)$ and $x \mapsto \int_{h(c)}^x f(t)\,dt$. Hence $\varphi'(s) = f(h(s))h'(s)$ by chain rule and Theorem 23. Again, by the same theorem, we have

$$\int_{h(c)}^{h(d)} f(t)\,dt = \varphi(d) - \varphi(c) = \int_c^d \varphi'(s)\,ds = \int_c^d f(h(s))h'(s)\,ds.$$

(Why does the integrand on the left most side make sense?)

□

Proposition 30. *Let $f\colon I \to \mathbb{C}$ be continuous. Then*

$$|\int_a^b f(t)dt| \le \int_a^b |f(t)|dt.$$

Proof. If f is real valued, then $-|f(t)| \le f(t) \le |f(t)|$ so that $\int -|f| \le \int f \le \int |f|$ Hence $|\int f| \le \int |f|$ in this case.

Let f be complex valued. Choose $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and $\alpha \int_a^b f(t)dt \in \mathbb{R}$. (If $\int_a^b f(t)\,dt = re^{it}$ is a polar representation, then we may take $\alpha = e^{-it}$.) By linearity, $\alpha \int_a^b f = \int_a^b \alpha f$ so that

$$\int_a^b \mathrm{Re}\,(\alpha f(t))\,dt = \mathrm{Re}\,(\alpha \int_a^b f(t)\,dt) = \alpha \int_a^b f(t)\,dt. \tag{1}$$

Also, observe that

$$|\mathrm{Re}\,(\alpha f(t)| \le |\alpha f(t)| = |\alpha||f(t)| = |f(t)|. \tag{2}$$

Hence

$$\begin{aligned}
|\int_a^b f(t)dt| &= |\alpha \int_a^b f(t)dt| && \text{(by our choice of } \alpha\text{)}\\
&= |\int_a^b \mathrm{Re}\,(\alpha f(t))dt| && \text{(by 1)}\\
&\le \int_a^b |\mathrm{Re}\,(\alpha f(t))|dt && \text{(by the real case)}\\
&\le \int_a^b |f(t)|dt && \text{(by 2 and monotonicity of the integral).}
\end{aligned}$$

This completes the proof. $\square$

Proposition 31. *If $f_n\colon [a,b] \to \mathbb{C}$ are continuous and they converge uniformly on $[a,b]$ to an $f\colon [a,b] \to \mathbb{C}$, then f is necessarily continuous and we have $\lim_n \int_a^b f_n(t)\,dt = \int_a^b f(t)\,dt$.*

Proof. Follows from the observation:

$$\begin{aligned}
|\int_a^b f_n(t)\,dt - \int_a^b f(t)\,dt| &= |\int_a^b [f_n(t) - f(t)]\,dt|\\
&\le \int_a^b |f_n(t) - f(t)|\,dt\\
&\le \int_a^b \varepsilon_n = \varepsilon_n(b-a),
\end{aligned}$$

where $\varepsilon_n := \sup\{|f_n(t) - f(t)| : t \in [a,b]\}$. Since $f_n \to f$ uniformly on $[a,b]$, $\varepsilon_n \to 0$. The result follows.(Question: Where did we use the fact that f is continuous?) $\square$

We give applications of this result which constitute some of the important results of Cauchy Theory in the next chapter.

Ex. 32. Let $f(z) := \sum_{n=0}^{\infty} a_n(z-z_0)^n$ for $z \in B(z_0, R)$. Then for $0 \le r \le R$, and $0 \le t \le 2\pi$, we have the Parseval identity:

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{it})|^2 \, dt = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}. \tag{3}$$

If $|f(z)| \le M(r)$ for $|z - z_0| = r$, then

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \le M(r)^2. \tag{4}$$

Hint: Observe that $\int_0^{2\pi} |s_n(t)|^2 \, dt = \sum_{k=0}^{n} |a_k|^2 r^{2k}$. Then use Ex. **??**.