Complex Analysis: Handout-3 (Cauchy-Riemann Equations)

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1 Cauchy-Riemann Equations

Now going back to the first definition and assuming that f is complex differentiable at z, are there any special directions we want to investigate the existence of the limit of the difference quotients? We want h to approach 0 either completely from the real axis or from the imaginary axis. We first look at $\frac{f(z+h)-f(z)}{h}$ as $h \in \mathbb{R}$ goes to 0. We have

$$\begin{aligned} f'(z) &= \lim_{h\in\mathbb{R},\, h\to 0} \frac{f(z+h)-f(z)}{h} \\ &= \lim_{h\in\mathbb{R}, h\to 0} \frac{[u(x+h,y)+iv(x+h,y)]-[u(x,y)+iv(x,y)]}{h} \\ &= u_x(x,y)+iv_x(x,y). \end{aligned} \tag{1}$$

We consider increments ih with $h \in \mathbb{R}$ and proceed in a similar way to get

$$f'(z) = \frac{1}{i}(u_y + iv_y) = -iu_y + v_y. \tag{2}$$

From (1) and (2), using the uniqueness of the limits, we get the Cauchy-Riemann equations:

$$u_x = v_y \qquad \text{and} \qquad u_y = -v_x. \tag{3}$$

In a similar way, the reader can show that if f is differentiable at z, then $\frac{\partial f}{\partial x}(z) = -i\frac{\partial f}{\partial y}(z) = f'(z)$. We use the "new variables" z and $\overline{z}$. Then $x = (z+\overline{z})/2$ and $y = (z-\overline{z})/2i$. Applying chain rule formally, we get

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} \Rightarrow \frac{\partial f}{\partial z} = [f_x - if_y]/2 \\ \frac{\partial f}{\partial \overline{z}} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}} \Rightarrow \frac{\partial f}{\partial \overline{z}} = [f_x + if_y]/2. \end{aligned}$$

This allows us to define two first order partial differential operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and } \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Apart from the Cauchy-Riemann equations in classical form

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{4}$$

we have them in a couple of other forms too:

$$f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \tag{5}$$

$$\frac{\partial f}{\partial \overline{z}} = 0. \tag{6}$$

Loosely speaking, (6) says that if f is holomorphic, then f, (though it is a function of x and y and hence a function of z and $\overline{z}$) is a function of z alone. In particular, if f is a polynomial in x and y iwth complex coefficients, it is a polynomial in z iff $\frac{\partial f}{\partial \overline{z}} = 0$.

Ex. 1. This is a compendium of standard applications of the Cauchy-Riemann equations (4). We assume that U is open and connected. Let $f\colon U \to \mathbb{C}$ be given.
(1) Assume that f is differentiable on U and that $f' = 0$ on U. Show that f is a constant using (4).
(2) If f is holomorphic on U and $\operatorname{Re} f$ (resp. $\operatorname{Im} f$) is a constant on U, then f is a constant.
(3) If f is holomorphic on U and assumes only real (resp. purely imaginary) values, then f is a constant.
(4) If f and $\overline{f}$ are both holomorphic on U, then f is a constant.
(5) If f is holomorphic on U and if $|f|$ is a constant, then f is a constant. *Hint:* Show that $\overline{f}$ is holomorphic.
(6) Let f be holomorphic on U. Assume that there exists a constant $\alpha \in \mathbb{R}$ such that $f(z) = |f(z)|e^{i\alpha}$ for all $z \in U$. Then f is a constant.

2 A Better way of Looking at C-R equations

Let $f\colon U \subset \mathbb{C} \to C$ be holomorphic. Write $f = u + iv$. Fix $z = x + iy \in U$. Let $f'(z) = \lambda = a + ib$. Note that $U \subset \mathbb{C}$ may also be considerd as a subset of $\mathbb{R}^2$. Observe that U is open in $\mathbb{C}$ iff it is open in $\mathbb{R}^2$. We claim that $u, v\colon U \subset \mathbb{R}^2 \to \mathbb{R}$ is a differentiable and we have $Du(x,y)(h,k) = ah - bk$ and $Dv(x,y)(h,k) = bh + ak$. If u is differentiable, we know that $Du(x,y)(h,k) = u_x h + u_y k$ etc, and hence we have $u_x = a = v_y$ and $u_y = -v_x$.

Proof. Let $\varepsilon > 0$ be given. Since f is holomorphic, for this $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |\xi| < \delta \implies |f(z+\xi) - f(z) - \lambda\xi| < \varepsilon. \tag{7}$$

We write $\xi = h + ik$. Then writing the real and imaginary part in (7), we get the desired results. □

Conversely, if $u, v\colon U \subset \mathbb{R}^2 \to \mathbb{R}$ are differentiable and satsify the C-R equation $u_x = v_y$ and $u_y = -v_x$, then$f = u + iv\colon U \subset \mathbb{C} \to \mathbb{C}$ is holomorphic. We also have $f'(x) = u_x - iu_y$. We let $a = u_x$ and $b = -u_y$.

This follows essentially tracing the steps. If $\varepsilon > 0$ is given, let $\delta > 0$ be chosen so that

$$|u(x+h,y+k) - u(x,y) - u_x h - u_y k| < \varepsilon \, \|(h,k)\|$$
$$|v(x+h,y+k) - v(x,y) - v_x h - v_y k| < \varepsilon \, \|(h,k)\|$$

this is rewritten, using C-R equations, as

$$|v(x+h,y+k) - v(x,y) + u_y h - u_x k| < \varepsilon \, \|(h,k)\| \, .$$

If write $\lambda = u_x - iu_y = a + ib$, then the equations above lead us to conclude that

$$\begin{aligned} &|f(z+\xi) - f(z) - \lambda\xi| \\ &= |(u(x+h,y+k) - u(x,y) - ah + bk) + i(u(x+h,y+k) - u(x,y) + bh + ak)| \\ &< \varepsilon|\xi| + \varepsilon|\xi|. \end{aligned}$$

Remark 2. The standard result in textbooks is: if the partial derivatives of u and v are continuous and satisfy the C-R equations, then $f = u + iv$ is holomorphic. The result is weaker than the one in the exercise. For, let $F = (u, v)$. Assume that u, v are continuously differentiable and satisfy the C-R equations, then F is continuously differentiable as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$ and u, v satisfy the C-R equations. But in Ex. 8 above, we need not have to assume that F is continuously differentiable. Thus, our formualtion is stronger than the conventional result.

3 A Sophisticated Way of Looking at C-R Equations

There is a geometric meaning behind the Cauchy-Riemann equations (3). This involves the "complex structure" on $\mathbb{R}^2$, i.e., the scalar multiplication by i on $\mathbb{C} \ni z = (x, y) \in \mathbb{R}^2$. The following exercises are in fact a series of remarks which will explicate this. (We chose not to write too much details, as it only smothers the real understanding.)

Ex. 3. Consider $\mathbb{C}$ as a vector space over itself. Show that the $\mathbb{C}$ (or complex) linear maps from $\mathbb{C}$ to $\mathbb{C}$ are of form $\varphi_\lambda(z) = \lambda z$ for a unique $\lambda \in \mathbb{C}$.

Ex. 4. Consider $\mathbb{C}$ as a vector space over $\mathbb{R}$. We fix once and for all $\{1, i\}$ as an ordered basis of $\mathbb{C}$ over $\mathbb{R}$. Show that with respect to to this basis, the matrix representation of φ_λ is $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ where $\lambda = a + ib$.

In particular, φ_i corresponds to $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Ex. 5. Conversely, a real linear map $\varphi \colon \mathbb{C} \to \mathbb{C}$ whose matrix representation $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is linear over $\mathbb{C}$ iff $AJ = JA$ (matrix multiplication). As a consequence, we have $a = d$ and $b = -c$ and hence $\varphi = \varphi_{a-ib}$. *Hint:* A is $\mathbb{C}$-linear iff $A(iz) = iA(z)$ for $z \in \mathbb{C}$.

Ex. 6. Let $z = x + iy \in \mathbb{C}$. Then $|z| = \|(x, y)\|$, the Euclidean norm on $\mathbb{R}^2$.

For $f: U \subset \mathbb{C} \to \mathbb{C}$ we let $f = u + iv$. Then $u, v: U \subset \mathbb{R}^2 \to \mathbb{R}$. We let $F(x, y) := (u(x, y), v(x, y))$ for $(x, y) \in U$. Note that under the real linear isomorphism $z = x + iy \mapsto (x, y)$, the basis element 1 (resp. i) goes to e_1 (resp. e_2).

Ex. 7. Let $f: U \subset \mathbb{C} \to \mathbb{C}$ be differentiable at $z \in U$. Then there exists $\lambda = a + ib$ (in fact $\lambda = f'(z)$) such that for $\varepsilon > 0$ there exists $\delta > 0$ with the property that $|h| < \delta$ implies that $|f(z + h) - f(z) - \varphi_\lambda(h)| < \varepsilon |h|$. This is equivalent to saying that

$$\left\| F((x, y) + (h_1, h_2)) - F(x, y) - \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\| < \varepsilon \left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|,$$

whenever $\left\|(h_1, h_2)^t\right\| < \delta$. Thus F is differentiable as map from $U \subset \mathbb{R}^2$ to $\mathbb{R}^2$ with total derivative $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

But if F is differentiable, then we know that its matrix representation with respect to the standard basis $\{e_1, e_2\}$ is given by its Jacobian matrix $\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$. Hence we deduce the C-R equations.

Ex. 8. Conversely, if $F = (u, v): U \subset \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at (x_0, y_0) with u and v satisfying the C-R equations, then the map $f(z) := u(x, y) + iv(x, y)$ for $z = x + iy$ is differentiable at $z_0 = x_0 + iy_0$ with $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$.