Complex Analysis: Handout-7 (Path Integrals)

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1 Path Integrals

Definition 1. Let γ be smooth path in U and $f\colon [\gamma] \to \mathbb{C}$ be continuous. We define the *integral* of f *over* γ by setting

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt.$$

If γ is just a path, using the above notation, we set

$$\int_\gamma f(z)\, dz := \sum_{j=0}^{n-1} \int_{t_j}^{t_{j+1}} f(\gamma(t))\gamma'(t)\, dt.$$

Note that the integrals on the RHS make sense, since γ' exists and is continuous on $[t_j, t_{j+1}]$. The complex number $\int_\gamma f$ is called the integral of f over the path γ.

Note also that if $\gamma(t) = t$, $t \in [a, b]$, then $\int_\gamma f(z)dz = \int_a^b f(t)dt$.

Show that $\int_\gamma \cos z\, dz = \cosh(1) - i \sinh(1)$ where γ is the line segment from $-\pi/2 + i$ to $\pi + i$.

Example 2. Let $\gamma, \sigma, \tau\colon [0, 2\pi] \to \mathbb{C}^*$ be given by $\gamma(t) = e^{it}$, $\sigma(t) = e^{2it}$ and $\tau(t) = e^{-it}$. Note that $[\gamma] = [\sigma] = [\tau]$. Let $f(z) = z^{-1}$ for $z \in \mathbb{C}^*$. We have $\int_\gamma f = 2\pi i$, $\int_\sigma f = 4\pi i$ and $\int_\tau f = -2\pi i$. Thus the path integral depends on the path and not on the trace. See also the next exercise.

Ex. 3. Compute the following path-integrals $\int_\gamma f(z)\, dz$:
(1) $f(z) := |z|^2$ and γ is the line segment from 2 to $3 + i$. Ans. $\frac{20(1+i)}{3}$.
(2) $f(z) := \mathrm{Re}\,(z)$ and γ is the line segment from 1 to $-i$. Ans. $\frac{-(1+i)}{2}$.

Theorem 4. *Let $f\colon U \to \mathbb{C}$ be continuous. Assume that there exists an $F\colon U \to \mathbb{C}$ such that $F' = f$ on U. Let $\gamma\colon [a, b] \to U$ be any path. Then*

$$\int_\gamma f(z)\, dz = F(\gamma(b)) - F(\gamma(a)).$$

In particular, if γ is closed, then $\int_\gamma f(z) = 0$. □

Corollary 5. *If $f: U \to \mathbb{C}$ is continuous and has a primitive in U, then $\int_\gamma f = 0$ for any closed path γ in U.* □

Example 6. Show that $\int_\gamma (z-a)^n dz$, where $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$ given by

$$\int_\gamma (z-a)^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1. \end{cases}$$

Ex. 7. Let $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$. For $n \in \mathbb{N}$, show that

$$\frac{1}{2\pi} \int_\gamma (2\cos\theta)^{2n}\, d\theta = \frac{(2n)!}{n!n!}.$$

Hint: Evaluate $\int_\gamma (z + \frac{1}{z})^{2n} \frac{dz}{z}$ using binomial expansion. Among the terms of the form $\binom{2n}{k} z^{2n-k} z^{-k}$, which is the term that will contribute to the integral? Look at Example 6.

Proposition 8. *Let γ be a path and $f: [\gamma] \to \mathbb{C}$ be continuous. Then $\int_{\tilde{\gamma}} f\, dz = -\int_\gamma f\, dz$.*

Proposition 9. *Let $f: U \to \mathbb{C}$ be continuous with a primitive F in U. Let γ_j, $j = 1, 2$, be paths in U both having the same initial (resp. terminal) points z and w respectively. Then $\int_{\gamma_1} f = F(w) - F(z) = \int_{\gamma_2} f$. Thus, in this case $\int_\sigma f$ depends only on the end points of σ for any path σ in U.*

Proposition 10 (ML Inequality)**.** *Let $\gamma: [a,b] \to \mathbb{C}$ be a path. Let $f: [\gamma] \to \mathbb{C}$ be continuous with $|f(\gamma(t))| \le M$ for all $t \in [a,b]$. We have*

$$|\int_\gamma f\, dz| \le ML(\gamma). \tag{1}$$

Ex. 11. Establish the following:
(1) $|\int_\gamma \frac{dz}{z^2+4}| \le \frac{\pi R}{(R^2-4)}$ where $\gamma(t) = Re^{it}$ for $0 \le t \le \pi$ and $R > 2$.
(2) $|\int_\gamma e^{-z}\, dz| \le 2$, where γ is the line segment from $-i$ to i.
(3) $|\int_\gamma \frac{dz}{z^4}| \le 4\sqrt{2}$ where $\gamma := [i, 1]$.
(4) $|\int_\gamma \frac{e^z}{z}\, dz| \le 2\pi e$ where γ is the unit circle with the standard parametrization.

Ex. 12. Let $f: \mathbb{C} \to \mathbb{C}$ be continuous and bounded. Let $\gamma_R(t) := Re^{it}$ for $0 \le t \le 2\pi$. Show that $\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{(z-z_0)^2}\, dz = 0$ for each z_0.

Proposition 13. *Let $f: U \to \mathbb{C}$ be continuous. Then f has a primitive in U iff $\int_\gamma f = 0$ for any closed path γ in U.*

Proposition 14. *Let $\gamma: [a,b] \to \mathbb{C}$ be a path. Let $f_n: [\gamma] \to \mathbb{C}$ be continuous ($n \in \mathbb{N}$) and that f_n converge uniformly on $[\gamma]$ to a function $f: [\gamma] \to \mathbb{C}$. Then $\int_\gamma f_n \to \int_\gamma f$.*

Proposition 15. *Let $\gamma(t) := z_0 + re^{it}$ for $0 \le t \le 2\pi$. Then*

$$\int_\gamma \frac{dz}{z-a} = \begin{cases} 2\pi i & a \in B(z_0, r) \\ 0 & a \notin B[z_0, r]. \end{cases} \tag{2}$$

Proof. Note that this is not Example 6. The trick here is to write

$$\frac{1}{z-a} = \frac{1}{(z-z_0)-(a-z_0)} = \frac{1}{(z-z_0)[1-\frac{a-z_0}{z-z_0}]} = \frac{1}{z-z_0}\frac{1}{1-w},$$

where $w := \frac{a-z_0}{z-z_0}$. If $a \in B(z_0, r)$, then $|w| < 1$ and we can substitute the geometric series expansion for $\frac{1}{1-w}$ in the integral. Since the convergence is uniform on $[\gamma]$, by Corollary **??**, we have

$$\begin{aligned}\int_\gamma \frac{1}{z-a}\,dz &= \int_\gamma \frac{1}{z-z_0}\sum_{n=0}^{\infty}(a-z_0)^n(z-z_0)^{-n} \\ &= \sum_{n=0}^{\infty}(a-z_0)^n \int_\gamma (z-z_0)^{-n-1} \\ &= 2\pi i,\end{aligned}$$

where we have used Example 6 to each of the terms of the series in the last but one line.

If $a \notin B[z_0, r]$ we then write $\frac{1}{z-a} = -\frac{1}{a-z_0}\frac{1}{1-w}$, where $w = \frac{z-z_0}{a-z_0}$. Proceeding as above, we get

$$\begin{aligned}\int_\gamma \frac{1}{z-a}\,dz &= \int_\gamma \frac{1}{a-z_0}\sum_{n=0}^{\infty}(a-z_0)^{-n}(z-z_0)^n \\ &= \sum_{n=0}^{\infty}(a-z_0)^{-n-1}\int_\gamma (z-z_0)^n \\ &= 0,\end{aligned}$$

again by Example 6. □

Theorem 16. *Let γ be any path and $f\colon [\gamma] \to \mathbb{C}$ be continuous. Then $F(z) := \int_\gamma \frac{f(w)}{w-z}\,dw$ is differentiable on $\mathbb{C} \setminus [\gamma]$. In fact, F is analytic in $\mathbb{C} \setminus [\gamma]$.*

Proof. The trick employed in the proof of the last proposition is implemented again.

Note that $\mathbb{C} \setminus [\gamma]$ is open.[1] If $z_0 \notin [\gamma]$, choose $r > 0$ such that $B(z_0, r) \cap [\gamma] = \emptyset$. If $z \in B(z_0, r/3)$, then $|\frac{z-z_0}{w-z_0}| < 1/2$ for $w \in [\gamma]$. See Figure **??**. Thus the series

$$\frac{f(w)}{w-z} = \frac{f(w)}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n$$

of functions in w for fixed z and z_0 converges uniformly on $[\gamma]$. Therefore,

$$F(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n,$$

where $a_n := \int_\gamma \frac{f(w)}{(w-z)^{n+1}}\,dw$. □

[1] We show that $[\gamma]$ is closed in $\mathbb{C}$. Let $p \in \mathbb{C}$ be a limit point of $[\gamma]$. Then there exist $t_n \in [a,b]]$ such that $\gamma(t_n) \to p$. Since $t_n \in [a,b]$, by Bolzano-Weierstrass theorem applied to $[a,b]$, (t_n) has a convergent subsequence, say (t_{n_k}), converging to $t_0 \in [a,b]$. Since γ is continuous at t_0, we deduce that $\gamma(t_{n_k}) \to \gamma(t_0)$. Hence $p = \gamma(t_0)$, by the uniqueness of the limit. We may also show that $[\gamma]$ is closed, by observing that $[\gamma]$ is compact.