Complex Analysis: Handout-7 (Properties of Holomorphic Functions)

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

Theorem 1 (Cauchy Integral Formula). *Let U be an open set star-shaped at $p \in U$. Let $f \in H(U)$. Let γ be a closed path in U. Then for any $a \in U \setminus [\gamma]$, we have*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-a}\,dw = f(a)\left[\frac{1}{2\pi i}\int_\gamma \frac{dw}{w-a}\right]. \tag{1}$$

Corollary 2. *Let U be any open set and $B(z_0, R) \subset U$. Let $f \in H(U)$. Let $\gamma(t) := z_0 + re^{it}$, $0 < r < R$ and $0 \le t \le 2\pi$. Then we have*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}, \qquad z \in B(z_0, r). \tag{2}$$

Theorem 3. *Any holomorphic function f on an open set U is an analytic function. In particular, f' is also holomorphic in U.* □

Theorem 4 (CIF for Derivatives). *Let $f \in H(U)$ and $B(z_0, R) \subset U$. Let $\gamma_r(t) := z_0 + re^{it}$ for $0 < r < R$. We have*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\gamma_r} \frac{f(w)}{(w-z)^{n+1}}\,dw. \tag{3}$$

In particular, we obtain the Cauchy's estimates for the derivatives:

$$|f^{(n)}(z_0)| \le n! r^{-n} M(r), \text{ where } M(r) := \sup\{|f(z)| : |z - z_0| = r\}. \tag{4}$$

Corollary 5. *Let $f \in H(U)$ and $B(a, R) \subset U$. Then the Taylor series of f converges to f absolutely and uniformly on compact subsets of $B(a, R)$.* □

Theorem 6 (Liouville's Theorem). *Let f be a bounded entire function on $\mathbb{C}$, say, $|f(z)| \le M$ for $z \in \mathbb{C}$. Then f is a constant.* □

Theorem 7. *Let U be a connected open set in $\mathbb{C}$, and let $f\colon U \to \mathbb{C}$ be holomorphic. Suppose that the set of zeros $Z(f) := \{z \in U : f(z) = 0\}$ has a cluster point* **in** *U. Then $f = 0$ on U.* □

Corollary 8 (Identity Theorem)**.** *Let U be connected and open. Let A be a subset of U which has a cluster point* **in** *U. Let $f, g\colon U \to \mathbb{C}$ be holomorphic. Assume that $f(z) = g(z)$ for all $z \in A$. Then $f = g$ on U.* □

Theorem 9 (Mean Value Property)**.** *Let $f \in H(U)$ and $B(a,R) \subset U$ and $\gamma_r(t) = a + re^{it}$, $0 < r < R$. Then*

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})\,dt. \tag{5}$$

Theorem 10 (Maximum Modulus Principle)**.** *Let $U \subset \mathbb{C}$ be a connected open set. Let $f \in H(U)$. Assume that there exists $a \in U$ such that $|f(a)| \geq |f(z)|$ for all $z \in B(a,R) \subset U$. Then f is a constant in U.* □

Corollary 11. *Let U be a bounded and connected open set in $\mathbb{C}$, $f \in H(U)$ and continuous on the closure $\overline{U}$ of U. Then either f is a constant or $|f|$ attains its maximum on the boundary of U.* □

Corollary 12 (Minimum Modulus Principle)**.** *If f is a nonconstant holomorphic function on a connected open set U then $z \in U$ cannot be a relative local minimum of $|f|$ unless $f(z) = 0$.* □

Theorem 13 (Open Mapping Theorem)**.** *The image of an open connected set U under a nonconstant holomorphic function is open.* □

Theorem 14 (Schwarz Lemma)**.** *Let $f\colon B(0,1) \to B(0,1)$ be holomorphic. Assume that $f(0) = 0$. Then*
(i) *$|f(z)| \leq |z|$ for all $z \in B(0,1)$ and*
(ii) *$|f'(0)| \leq 1$.*

Furthermore, equality holds in (i) for some nonzero z or equality holds in (ii) iff $f(z) = cz$ for some c with $|c| = 1$. □

Theorem 15 (Weierstrass)**.** *Let $f_n \in H(U)$ for $n \in \mathbb{N}$. Assume that f_n converges to f uniformly on compact subsets of U. Then $f \in H(U)$. Furthermore, for any $k \in \mathbb{N}$, $f_n^{(k)}$ converges to $f^{(k)}$ uniformly on compact subsets of U.* □