Complex Analysis: Handout-8 (Weierstrass Theorem)

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Theorem 1 (Weierstrass)**.** *Let $f_n \in H(U)$ for $n \in \mathbb{N}$. Assume that f_n converges to f uniformly on compact subsets of U. Then $f \in H(U)$. Furthermore, for any $k \in \mathbb{N}$, $f_n^{(k)}$ converges to $f^{(k)}$ uniformly on compact subsets of U.*

Proof. First of all note that f is continuous on U. Let $z \in U$. Choose $r > 0$ such that $B(z, 2r) \subset U$. Then the closed ball $B[z, r]$ is a closed and bounded subset of $\mathbb{C}$ and hence is compact. Hence $f_n \to f$ uniformly on $B[z, r]$. Hence f is continuous on $B[z, r]$ and in particular at z.

Let $z_0 \in U$ and $r > 0$ be such that $B[z_0, r] \subset U$. Let $z \in B(z_0, r)$. Let $\gamma(t) := z_0 + re^{it}$, $0 \leq t \leq 2\pi$. By (**??**), we have $f_n(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(w)}{w-z}\, dw$. Since $[\gamma]$ is compact, and $z \notin [\gamma]$, we have $\varepsilon := d(z, [\gamma]) > 0$.

Since $[\gamma]$ is compact, $f_n \to f$ uniformly on $[\gamma]$. Also, the function $(w - z)$ is bounded away from zero: $|w - z| \geq \varepsilon$ so that if $g := \frac{1}{(w-z)}$, then $|g| \leq 1/\varepsilon$, and hence g is a bounded function. It is now easy to see that $f_n g \to fg$ uniformly on $[\gamma]$. Hence $\frac{f_n(w)}{w-z} \to \frac{f(w)}{w-z}$ uniformly on $[\gamma]$

Hence $f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(w)}{w-z}\, dw$. Now, f is holomorphic on $B(z_0, r)$ by earlir theorems.

As argued above, we see that

$$\frac{f_n(w)}{(w-z)^{k+1}} \to \frac{f(w)}{(w-z)^{k+1}},$$

uniformly on $[\gamma]$. Hence

$$\frac{n!}{2\pi i} \int_\gamma \frac{f_n(w)}{(w-z)^{k+1}} \to \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w-z)^{k+1}}.$$

From CIF for derivatives, it follows that $f_n^{(k)}(z) \to f^{(k)}(z)$ for every $z \in B(z_0, r)$. However the uniform convergence is not established. So we argue more carefully as follows.

In view of CIF for derivatives, we have

$$f_n^{(k)}(z) - f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f_n(w) - f(w)}{(w-z)^{k+1}}. \qquad z \in B(z_0, r).$$

If $z \in B[z_0, \rho]$ with $\rho < r$, (for example, $\rho = r/2$) then we have

$$|f_n^{(k)}(z) - f^{(k)}(z)| \leq \frac{k!}{2\pi} \max_{z \in \gamma} |f_n(w) - f(w)| \frac{2\pi r}{(r-\rho)^{k+1}} \to 0 \text{ as } n \to \infty.$$

Thus $f_n^{(k)}$ converges to $f^{(k)}$ uniformly on closed disks $B[z_0, r]$.

We claim that $f_n^{(k)}$ converges to $f^{(k)}$ uniformly on compact subsets of U.

If K is compact subset of U, then there exist $z_i \in K$ and $r_i > 0$, $1 \leq i \leq n$, such that $B[z_i r_i] \subset U$ and such that $K \subset \cup_{i=1}^n B(z_i, r_i)$. Note that if $f_n \to f$ uniformly on A and B, so does it on $A \cup B$. Since we have already shown that $f_n \to f$ uniformly on each of $B[z_i, r_i]$, the result follows.

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