Connected and Path-connected Spaces

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The aim of this article is to introduce the readers to an easier way of working with connectedness concept. If the reader's background does not include general (abstract) topological spaces, he may assume that the spaces are metric spaces.

1 Connectedness

Definition 1. A topological space X is said to be *connected*, if the only subsets of X which are both open and closed are the empty set $\emptyset$ and X. In other words, a topological space is connected whenever a subset A is both open and closed in X, then either $A = \emptyset$ or $A = X$.

A subset A of a topological space X is said to be connected if A is a connected space when considered as a topological space with the induced (or subspace) topology. In the case of metric space (X, d), this amounts to saying that (A, δ) is connected, where δ is the restriction of the metric d on X to A.

Therefore, if a topological space X is not connected, there will be a proper non-empty sub set A of X which is both open and closed in X. If A is a proper non-empty sub set of X and both open and closed, then $B = A^c$, its complement is also a proper non-empty sub set of X which is both open and closed in X. In other words a topological space X is not connected iff there exist two disjoint proper non-empty sub sets A and B such that A and B are both open and closed in X and $X = A \cup B$. In such case we also say that the pair (A, B) is a *disconnection* of X.

Example 2. Let X be a set such that $\mid X \mid \geq 2$ with discrete topology (or discrete metric). Then X is not connected.

Example 3. The subset $\{\pm 1\} \subset \mathbb{R}$ with the subspace topology is not connected. (Why?) In the sequel, we consider $\{\pm 1\}$ as a subset of $\mathbb{R}$.

Now we prove a single most important theorem in connectedness which supplies us an abundance of examples of connected and non-connected spaces.

Theorem 4. *A topological space X is connected iff every continuous function $f \colon X \to \{\pm 1\}$ is a constant function.*

Proof. Let X be a connected space and $f\colon X \to \{\pm 1\}$ a continuous function. We want to show that f is a constant function. If f is non-constant, then it is on-to. Let $A = f^{-1}(1)$ and $B = f^{-1}(-1)$. Then A and B are disjoint non-empty subsets of X such that A and B are both open and closed subsets of X and $X = A \cup B$.(Why?). This is a contradiction. Therefore f is constant.

Conversely, let us assume that X is not connected. Therefore there exist two disjoint proper non-empty subsets A and B in X such that A and B are both open and closed in X and $X = A \cup B$. Now we define a map $f\colon X \to \{\pm 1\}$ as

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ -1 & \text{if } x \in B \end{cases}$$

Then $f\colon X \to \{\pm 1\}$ is a continuous non-constant function. (Why?). This completes the proof. ☐

We shall now use this theorem to get examples of connected spaces.

Example 5. A set $J \subseteq \mathbb{R}$ is connected iff J is an interval.

Proof. Let J be a connected sub set of $\mathbb{R}$. Let us assume that J is not an interval. This means that there exist points $a < b$ in J and $c \in \mathbb{R}$ such that $a < c < b$ but $c \notin J$. Now we define a map $f\colon J \to \{\pm 1\}$ as

$$f(x) = \begin{cases} 1 & \text{if } x < c \\ -1 & \text{if } x > c \end{cases}$$

Now we claim that f is a continuous function. We need only to check that $f^{-1}(1)$ and $f^{-1}(-1)$ are open in J. By our definition $f^{-1}(1) = J \cap (-\infty, c)$ and $f^{-1}(-1) = J \cap (c, \infty)$ which are open, proper and nonempty subsets in J. (Why?) This is a contradiction to the fact that J is connected. Therefore for every pair of points a and b in J such that $a < b$, all the points c such that $a < c < b$ are also in J. This means that J is an interval in $\mathbb{R}$.

Conversely, let us assume that J is an interval in $\mathbb{R}$. Let $f\colon J \to \{\pm 1\}$ be a continuous function. We need to show that f is constant. If not, then there exist $a, b \in J$ such that $f(a) = 1$ and $f(b) = -1$. Since $a, b \in J$ and J is an interval, $[a, b] \subset J$. Hence, by applying the intermediate value theorem to the restriction f to $[a, b]$, there exists $c \in (a, b) \subset J$ such that $f(c) = 0$. This is a contradiction, since the codomain is $\{\pm 1\}$. ☐

Example 6. Let $\mathrm{M}(2, \mathbb{R})$ denote the set of all 2×2 matrices of real numbers. and $\mathrm{G}L(2, \mathbb{R}) := \{A \in \mathrm{M}(2, \mathbb{R}) : \det(A) \neq 0\}$ is not connected.

Proof. Here we identify $M(2, \mathbb{R})$ with $\mathbb{R}^4$ via the map $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a, b, c, d) \in \mathbb{R}^4$. Let $f\colon \mathrm{GL}(2, \mathbb{R}) \to \mathbb{R}$ be defined by $f(A) := \det(A)$. Complete the proof. ☐

Example 7. $\mathrm{O}(2, \mathbb{R}) := \{A \in \mathrm{GL}(2, \mathbb{R}) : AA^t = Id\}$ is not connected.

Proof. The equation $AA^t = Id$ shows that $\det(A) = \pm 1$ for every $A \in \mathrm{O}(2, \mathbb{R})$. This suggests us that we define the map $f\colon \mathrm{O}(2, \mathbb{R}) \to \{\pm 1\}$ by $f(A) := \det(A)$. Complete the proof. ☐

Proposition 8. *Let X be a topological space. Let A and B be two connected subsets of X such that $A \cap B \neq \emptyset$. Then $A \cup B$ is connected.*

Proof. Let $f \colon A \cup B \to \{\pm 1\}$ be a continuous function. We have to show that f is constant. Let $c \in A \cap B$. Since A is connected, the function $f \mid_A \colon A \to \{\pm 1\}$ is constant so that $f(a) = f(c)$ for all $a \in A$. Similarly, $f(b) = f(c) = 1$ for all $b \in B$. Thus $f(x) = f(c)$ for all $x \in A \cup B$. i.e., f is a constant. □

Proposition 9. *Let A be a connected subset of a space X. Let $A \subset B \subset \overline{A}$. Then B is connected.*

Proof. Let $f \colon B \to \{\pm 1\}$ be a continuous function. Without loss of generality, let us assume that $f = 1$ on A. Let $x \in B$. Since $\{f(x)\}$ is open in $\{\pm 1\}$, the set $U := f^{-1}(f(x))$ is an open containing x. Hence, there exists a point $a \in A \cap U$. Since $a, x \in U$ and $f = f(x)$ on U, it follows that $f(x) = f(a) = 1$. Thus $f = 1$ on B. □

Proposition 10. *Let $\{A_i : i \in I\}$ be a collection of connected subsets of a space X with the property that for all $i, j \in I$ we have $A_i \cap A_j \neq \emptyset$. Then $A := \cup_i A_i$ is connected.*

Proof. Fix A_i. Let $f \colon \cup A_j \to \{\pm 1\}$ be continuous. Since A_i is connected, f is a constant on it, say, $f = 1$ on A_i. Let $x \in A$. Then $x \in A_j$ for some j. Let $y \in A_i \cap A_j$. Then $f(x) = f(y)$ since A_j is connected and $x, y \in A_j$. Since $y \in A_i$, we have $f(y) = 1$. Hence for all $x \in A$, we conclude $f(x) = 1$. Hence A is connected. □

Proposition 11. *Let X be a connected topological space and $g \colon X \to Y$ be a continuous map. Then $g(X)$ is connected.*

Proof. We will show that any continuous map $f \colon g(X) \to \{\pm 1\}$ is constant.

Let $f \colon g(X) \to \{\pm 1\}$ be a continuous map. Then the map $f \circ g \colon X \to \{\pm 1\}$ is continuous. (Why?). Since X is connected, it follows that $g \circ f$ is constant. Hence f is constant. For, otherwise, there exist $y_1, y_2 \in g(X)$ such that $f(y_1) \neq f(y_2)$. Since $y_j \in g(X)$, this implies the existence of $x_j \in X$ such that $g \circ f(x_1) \neq g \circ f(x_2)$. In particular, $g \circ f$ is not a constant. Hence we are forced to conclude that f is constant. Thus $g(X)$ is connected. □

Corollary 12. *In the above proposition, if the map g is onto then Y is connected.*

Ex. 13. Show that the set $GL(2, \mathbb{R})$ is not connected.

Ex. 14. Show that the circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is connected.

Ex. 15. Show that the set $SO(2, \mathbb{R}) := \{A \in O(2, \mathbb{R}) : \det A = 1\}$ is connected. *Hint:* Write down all elements of $SO(2, \mathbb{R})$ explicitly.

Proposition 16. *Let X and Y be connected spaces. Then the product space $X \times Y$ is connected.*

Proof. Let $f: X \times Y \to \{\pm 1\}$ be a continuous map. Let $(x_0, y_0) \in X \times Y$ be fixed. Let (x, y) be an arbitrary point in $X \times Y$. If we show that $f((x, y)) = f((x_0, y_0))$, we are through.

To prove the above claim, let us first observe that for every point $y \in Y$, the map $i_y: X \to X \times Y$ defined by $i_y(x) := (x, y)$ is continuous; similarly the map $i_x: Y \to X \times Y$ defined by $i_x(y) := (x, y)$ is continuous for every point x in X. Therefore for every point y in Y, the subset $X \times \{y\} := \{(x, y) : x \in X\}$ is a connected subset of $X \times Y$; similarly, the subset $\{x\} \times Y := \{(x, y) : y \in Y\}$ is a connected subset of $X \times Y$ for every point x in X.

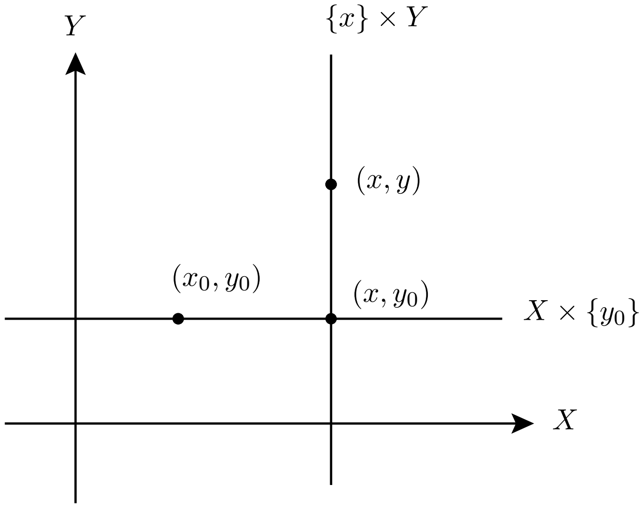

Figure 1: Connectedness of the product

Now the point (x, y_0) lies in both sets $X \times \{y_0\}$ and $\{x\} \times Y$. The restrictions of f to either of these sets are continuous and hence constants. We see that $f(x_0, y_0) = f(x, y_0))$ for all $x \in X$ and similarly, $f(x, y) = f(x, y_0)$ for all $y \in Y$. In particular, $f(x, y) = f(x, y_0) = f(x_0, y_0)$. (See Figure 1.) ☐

The following is a **typical way** in which connectedness hypothesis is used.

Theorem 17. *Let X be connected. Let $f: X \to \mathbb{R}$ be a locally constant function, i.e., for each $x \in X$, there exists an open set U_x containing x with the property that f is a constant on U_x. Then f is a constant on X.*

Proof. First of all note that any locally constant function is necessarily continuous.

Fix $x_0 \in X$. We show that $f(x) = f(x_0)$ for all $x \in X$. Consider the set $E := \{x \in X \mid f(x) = f(x_0)\}$. As $x_0 \in E$, we see that E is nonempty. Since $E = f^{-1}(f(x_0))$, E is the inverse image of a closed set under the continuous map f and hence is closed.

If $x \in E$, since f is locally constant, there exists an open set U_x with $x \in U_x$ and f is constant on U_x. Thus for each $y \in U_x$, we have $f(y) = f(x)$. Since $x \in E$, we have $f(x) = f(x_0)$. Hence it follows that $f(x) = f(x_0)$ for all $x \in U_x$. In other words, $U_x \subset E$. Hence E is open. Thus E is nonempty, open and closed subset of the connected space X. Hence we must have $E = X$. ☐

As an immediate corollary we have

Theorem 18. *Let U be an open connected subset of $\mathbb{R}^n$ and $f: U \to \mathbb{R}$ be a differentiable function such that $Df(p) = 0$ for all $p \in U$. Then f is a constant function.*

Proof. To prove this theorem we will only use only the following fact which follows from mean value theorem. *Let U be an open convex subset of $\mathbb{R}^n$ and $f: U \to \mathbb{R}$ be a differentiable function such that $Df(p) = 0$ for all $p \in U$. Then f is constant on U.*

Now let f be as in the theorem. Then for each $x \in U$, since U is open, there exists an open ball $B(x, r_x) \subset U$. It is easy to see that any ball in $\mathbb{R}^n$ is convex. Thus an application of the calculus result quoted above shows us that f locally constant. □

2 Path Connected spaces

Definition 19. 1. Let X be a topological space. A continuous map $\gamma: [0,1] \to X$ is called a *path* in X. If $\gamma(0) = x$ and $\gamma(1) = y$, then γ is also called a path joining the points x and y or simply a path from x to y.

2. A topological space X is said to be path connected if for all points x and y in X, there exists a path $\gamma: [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

2.1 Examples & Exercises

Example 20. The space $\mathbb{R}^n$ is path connected. Any two points can be joined by a line segment: $\gamma(t) := x + t(y - x)$, for $0 \leq t \leq 1$. We call this path γ a linear path.

Example 21. For every $r > 0$, the circle $C_r := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$ is path connected.(Why?)

Example 22. The set $\{(x,y) \in \mathbb{R}^2 : x \geq 0 \quad \& \quad x^2 - y^2 = 1\}$ is path connected. Draw the picture and see that it is the "right" hand of the hyperbola $x^2 - y^2 = 1$. Similarly the left hand of a hyperbola is also path connected. However the hyperbola is not path connected. (Why?)

Example 23. The parabola $\{(x,y) \in \mathbb{R}^2 : y^2 = x\}$ is path connected.

Example 24. The union of the two parabolas $\{(x,y) \in \mathbb{R}^2 : y^2 = x\}$ and $\{(x,y) \in \mathbb{R}^2 : y = x^2\}$ is path connected.

Example 25. The union of the parabolas $\{(x,y) \in \mathbb{R}^2 : y^2 = x\}$ and $\{(x,y) \in \mathbb{R}^2 : y^2 = -x\}$ is path connected.

Example 26. The set $S^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$ is path connected. Let X and Y be two points in S^2. Then define $\gamma: [0,1] \to S^2$ by $\gamma(t) := \frac{X + t(Y-X)}{\|X + t(Y-X)\|}$. Then check that this gives us a path from X to Y. (Does it?).

Proposition 27. *Let X be a topological space. Let $\gamma_1: [0,1] \to X$ and $\gamma_2: [0,1] \to X$ be two paths such that $\gamma_1(1) = \gamma_2(0)$. Then there exists a path $\gamma_3: [0,1] \to X$ such that $\gamma_3(0) = \gamma_1(0)$ and $\gamma_3(1) = \gamma_2(1)$.*

Proof. Define the map $\gamma_3\colon [0,1] \to X$ such that

$$\gamma_3(t)\colon = \begin{cases} \gamma_1(2t) & \text{if} \quad t \le \frac{1}{2} \\ \gamma_2(2t-1) & \text{if} \quad t \ge \frac{1}{2} \end{cases}$$

Now we leave it as an exercise to verify that γ_3 is a path in X meeting our requirements. (Draw pictures and see geometrically). □

Proposition 28. *Let X be path connected. Then X is connected.*

Proof. Let $f\colon X \to \{\pm 1\}$ be a continuous function. We need to show that f is constant.

Let $x \neq y$ be two points in X. Since X is path connected, there exists a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Now, the map $f \circ \gamma\colon [0,1] \to \{\pm 1\}$ is continuous. Since $[0,1]$ is connected, the map $f \circ \gamma$ is constant. Therefore f is constant. (Why?). This proves that X is connected. □

The converse is not always true. However, in the case of open subsets of $\mathbb{R}^n$, the converse is also true and we prove this in

Theorem 29. *Let U be an open connected subset of $\mathbb{R}^n$. Then U is path connected.*

Proof. Let x_0 be a point in U and let

$$E := \{x \in U : \text{ there exists a path } \gamma \text{ such that } \gamma(0) = x \;\&\; \gamma(1) = x_0\}.$$

We will show that the set E is non-empty, both open and closed in U. Then since U is connected, it will follow that $E = U$ and this will prove the theorem. (Why?)

First we note that the set E is non-empty. The map $\gamma\colon [0,1] \to X$ defined by $\gamma(t) = x_0$ for all t is a path in X. Therefore x_0 is in E. Let x be a point in E. Since U is open there exists $r > 0$ such that $B(x,r) \subseteq U$. Let y be a point in $B(x,r)$. Since $B(x,r)$ is convex, there exists a linear path, say, γ_1, joining the points y and x. Since x is in E there exists a path γ_2 from x to the point x_0. From Proposition 27, it follows that there exists a path γ_3 from y to x_0. This means that $B(x,r) \subseteq E$. Hence E is open.

We will now show that E is also closed in U. Let $x \in U$ be a limit point of E. Therefore there exists a sequence x_n of points in E such that the sequence x_n converge to the point x. Since U is open there exists an $r > 0$ such that the open ball $B(x,r) \subseteq U$. Since the sequence x_n converges to the point x, there exists N in $\mathbb{N}$ such that the points $x_n \in B(x,r)$ for all $n \ge N$. Let γ_1 be the linear path from x to the point x_N and γ_2 be a path from x_N to x_0. From Proposition 27, there exists a path γ_3 from x to x_0. This means that the point x is in E. Hence E is closed and therefore $E = U$. □