Construction of Real Numbers

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Definition 1. Let $\mathbb{Q}$ be the set of all rational numbers. A sequence $(x_n), x \in \mathbb{Q}$ is said to be Cauchy if for every $\varepsilon \in \mathbb{Q}$, there exists a positive integer n_0 such that $|x_n - x_m| < \varepsilon$ for all $n, m \geq n_0$.

Examples: $(\frac{1}{n}), (1 + \frac{1}{n}), (\frac{2}{n^2})$ etc.

Definition 2. A sequence (x_n) in $\mathbb{Q}$ is said to be convergent **in** $\mathbb{Q}$ to a rational number a if for every $\varepsilon^+ \in \mathbb{Q}$ there exists a $n_0 \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for all $n \geq n_0$. In this case we write $\lim x_n = a$.

Notations: Let $\mathcal{C}$ be the set of all Cauchy sequences in $\mathbb{Q}$ and $\mathcal{N}$ be the set of all $(x_n) \in \mathcal{C}$ such that $\lim x_n = 0$. Elements of $\mathcal{N}$ are called null sequences.

We define addition and multiplication of two Cauchy sequences (x_n) and (y_n) in $\mathcal{C}$ as follows: $(x_n) + (y_n) := (x_n + y_n)$ and $(x_n) \cdot (y_n) := (x_n y_n)$.

Ex. 3. Prove that if $(x_n), (y_n) \in \mathcal{C}$ then $(x_n + y_n)$ and $(x_n y_n)$ are also in $\mathcal{C}$.

Lemma 4. *$\mathcal{C}$ is a commutative ring under the addition and multiplication defined above.*

Proof. Proof of this lemma is a routine checking. Note that $(0) = (0, 0, \ldots)$ and $(1) = (1, 1, \ldots)$ are the zero and the identity elements in $\mathcal{C}$. □

Definition 5. If R is a ring then a non empty subset $I \subseteq R$ is said to be an ideal of R if for all $x, y \in I$ and $r \in R$, $x + y \in I$ and $rx \in I$.

Lemma 6. *$\mathcal{N}$ is an ideal of $\mathcal{C}$.*

Proof. Let $(x_n), (y_n) \in \mathcal{N}$ and $(r_n) \in \mathcal{C}$. To prove that $(x_n + y_n)$ and $(r_n x_n) \in \mathcal{N}$. Using algebra of limits, $\lim (x_n + y_n) = \lim x_n + \lim y_n = 0 + 0 = 0$. Hence $(x_n + y_n) \in \mathcal{N}$. Next we prove that $(r_n x_n) \in \mathcal{N}$. Since (r_n) is a Cauchy sequence, (r_n) is bounded.(why?) That is, there exists $M \in \mathbb{Q}^+$ such that $|r_n| \leq M$, for all n. On the other hand $(x_n) \in \mathcal{N}$, therefore, for $\varepsilon \in \mathbb{Q}^+$, there exists $n_0 \in \mathbb{N}$ such that $|x_n| < \frac{\varepsilon}{M}$, for all $n \geq n_0$. Thus, $|r_n x_n| = |r_n||x_n| \leq M|x_n| < M\frac{\varepsilon}{M} = \varepsilon$, for all $n \geq n_0$. Hence $r_n x_n \in \mathcal{N}$. □

Definition 7. We define a relation $\sim$ on $\mathcal{C}$ as follows: for $(x_n), (y_n) \in \mathcal{C}$, we say that $(x_n) \sim (y_n)$ iff $(x_n) - (y_n) = (x_n - y_n) \in \mathcal{N}$. (Check that this is an equivalence relation on $\mathcal{C}$.) We define equivalence classes $\mathcal{C}/\mathcal{N} = \{(x_n) + \mathcal{N} \mid (x_n) \in \mathcal{C}\}$ as the set of real numbers denoted by $\mathbb{R}$. Note that $\mathcal{C}/\mathcal{N}$ is the quotient set in $\mathcal{C}$ w.r.t. the equivalence relation $\sim$.

We now make $\mathcal{C}/\mathcal{N}$ into a ring. This is a very common result in algebra if R is a ring and I is an ideal of R, then R/I can be made into a ring. For those who is not familiar with this result we define addition and multiplication in $\mathcal{C}/\mathcal{N}$ as follows: for $(x_n)+\mathcal{N}, (y_n)+\mathcal{N} \in \mathcal{C}/\mathcal{N}$,

$$((x_n)+\mathcal{N}) + ((y_n)+\mathcal{N}) := (x_n+y_n)+\mathcal{N} \text{ and } ((x_n)+\mathcal{N}) \cdot ((y_n)+\mathcal{N}) := (x_n y_n)+\mathcal{N}.$$

First of all we must check that these are well defined. This follows from the following general result from algebra.

Lemma 8. *Let R be a commutative ring and I be an ideal in R. Let R/I, the quotient of R w.r.t. the equivalence relation $\sim$ on R as: $x \sim y$ iff $x-y \in I$. If $x \sim x_1$ and $y \sim y_1$ then $x+y+I = x_1+y_1+I$ and $xy+I = x_1y_1+I$.*

Proof. Since $x \sim x_1, x-x_1 \in I$. Similarly $y-y_1 \in I$. Hence $x-x_1+y-y_1 = (x+y)-(x_1+y_1) \in I$. This implies that $x+y+I = x_1+y_1+I$. Next, $xy - x_1y_1 = xy - xy_1 + xy_1 - x_1y_1 = x(y-y_1)+(x-x_1)y_1 = x(y-y_1)+y_1(x-x_1) \in I$. This implies that $xy+I = x_1y_1+I$. □

Lemma 9. *Let $(x_n) \in \mathcal{C} \setminus \mathcal{N}$. There exists $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that $|x_n| > \varepsilon$, for all $n \geq n_0$. In fact, there exists $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that only one of the following is true.*

1. *Either $x_n \geq \varepsilon$, for all $n \geq n_0$, or*
2. *$x_n \leq -\varepsilon$, for all $n \geq n_0$.*

Proof. Since $(x_n) \in \mathcal{C} \setminus \mathcal{N}$, $(x_n) \notin \mathcal{N}$. Therefore there exists $\varepsilon > 0$ in $\mathbb{Q}$ such that for each $k \in \mathbb{N}$, there exists x_{n_k} such that $|x_{n_k}| > 2\varepsilon$. That is, either $x_{n_k} \geq 2\varepsilon$, or $x_{n_k} \leq -2\varepsilon$. But $(x_n) \in \mathcal{C}$, therefore, for the above ε there exists $N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$, for all $n, m \geq N$. Fix $k \in \mathbb{N}$ such that $n_k \geq N$. Then, for all $n, m \geq N$, $|x_n - x_m| < \varepsilon$. Or, in otherwords $x_n \in (x_{n_k} - \varepsilon, x_{n_k} + \varepsilon)$, for all $n \geq n_k \geq N$.

If $x_{n_k} \leq -2\varepsilon$. Then $x_{n_k} + \varepsilon \leq -\varepsilon$, hence $x_n \leq x_{n_k} + \varepsilon \leq -\varepsilon$, for all $n \geq n_k \geq N$.

If $x_{n_k} \geq 2\varepsilon$, then $x_{n_k} - \varepsilon \geq \varepsilon$. Hence $x_n \geq x_{n_k} - \varepsilon \geq \varepsilon$, for all $n \geq n_k \geq N$. If we take $n_0 = N$, the result follows. □

Theorem 10. *$\mathbb{R} = \mathcal{C}/\mathcal{N}$ is a field.*

Proof. It is easy to show that $\mathbb{R}$ is a commutative ring with the zero element $\mathcal{N}$ and the identity element $1+\mathcal{N}$. We need to check that if $x+\mathcal{N} \in \mathcal{C}/\mathcal{N}$ and $x \notin \mathcal{N}$ then it is invertible. That is, there exists $y+\mathcal{N}$ such that $(x+\mathcal{N}) \cdot (y+\mathcal{N}) = 1+\mathcal{N}$.

Let $x+\mathcal{N} \in \mathcal{C}/\mathcal{N}$ and $x \notin \mathcal{N}$. By Lemma 4 there exists $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $x_n > \varepsilon$, for all $n \geq N$. Define $y = (y_1, y_2, \ldots, y_N, 0, \ldots)$ such that $x_i + y_i \neq 0$, for $1 \leq i \leq N$. Note that $x+\mathcal{N} = (x+y)+\mathcal{N}$. Define $(x+y)^{-1} = (\frac{1}{x_1+y_1}, \ldots, \frac{1}{x_N+y_N}, \ldots)$. We claim that $(x+y)^{-1} \in \mathcal{C}$. Let $\delta \in \mathbb{Q}^+$ be given. For all $n, m \geq N$

$$|\frac{1}{x_n} - \frac{1}{x_m}| = \frac{|x_m - x_n|}{|x_n||x_m|} < \frac{|x_m - x_n|}{\varepsilon^2}.$$

Since $(x_n) \in \mathcal{C}$, for the above δ there exists $n_1 \in \mathbb{N}$ such that $|x_m - x_n| < \delta\varepsilon^2$, for all $n, m \geq n_1$. Choose $n_0 = \max(N, n_1)$, then $|\frac{1}{x_n} - \frac{1}{x_m}| < \frac{\delta\varepsilon^2}{\varepsilon^2} = \delta$, for all $n, m \geq n_0$. Hence $(x+y)^{-1} \in \mathcal{C}$. Since $x+\mathcal{N} = (x+y)+\mathcal{N}$, $(x+\mathcal{N}) \cdot ((x+y)^{-1}+\mathcal{N}) = ((x+y)+\mathcal{N}) \cdot ((x+y)^{-1}+\mathcal{N}) = 1+\mathcal{N}$. Hence $x+\mathcal{N}$ is invertible. □

Definition 11. An ideal I of a ring R is said to be a maximal ideal if J is an ideal containing I properly, then $J = R$.

Remark 12. In fact we have proved that $\mathcal{N}$ is a maximal ideal of $\mathcal{C}$.

Proof. Let I be an ideal of $\mathcal{C}$ such that $\mathcal{N}$ is properly contained in I. Let $x \in I \setminus \mathcal{N}$. By Theorem 2 there exists y such that $(x + \mathcal{N}) \cdot (y + \mathcal{N}) = 1 + \mathcal{N}$. Hence $xy + \mathcal{N} = 1 + \mathcal{N}$. That is, $1 - yx \in \mathcal{N} \subseteq I$. Since $x \in I$, $yx \in I$. This implies that $1 = 1 - yx + yx \in I$. Hence $I = \mathcal{C}$. □

Definition 13. A Cauchy sequence (x_n) in $\mathbb{Q}$ is said to be positive if there exists $\varepsilon \in \mathbb{Q}^+$ and $N \in \mathbb{N}$ such that $x_n > \varepsilon$, for all $n \geq N$.

Definition 14. A real number $\alpha \in \mathbb{R}$ is said to be positive, if $(x_n) \in \alpha$, then (x_n) is a positive sequence in $\mathbb{Q}$.

We need to check that this definition is well defined, that is, if $(x_n), (y_n) \in \alpha$ such that (x_n) is a positive sequence in $\mathbb{Q}$, then (y_n) is also a positive sequence in $\mathbb{Q}$.

Proof. Since (x_n) is a positive sequence in $\mathbb{Q}$, there exists $\varepsilon \in \mathbb{Q}^+$ and $n_1 \in \mathbb{N}$ such that $x_n > 2\varepsilon$, for all $n \geq n_1$. Also $(x_n - y_n) \in \mathcal{N}$, so for the above ε there exists $n_2 \in \mathbb{N}$ such that $|x_n - y_n| < \varepsilon$, for all $n \geq n_2$. Choose $n_0 = \max(n_1, n_2)$. So, $y_n = (y_n - x_n) + x_n > \varepsilon$, for all $n \geq n_0$. Hence (y_n) is also a positive sequence in $\mathbb{Q}$. □

Theorem 15. *(1) If (x_n) is a positive sequence in $\mathcal{C}$ and $(z_n) \in \mathcal{N}$, then $(x_n + z_n)$ is a positive sequence in $\mathcal{C}$.*
(2) If (x_n) and (y_n) are positive sequences is $\mathcal{C}$ then (x_n+y_n) and $(x_n y_n)$ are positive sequences in $\mathcal{C}$.

Proof. Proof of 1 is essentially the proof given for the well definedness of the above definition. So we leave this for the reader to complete.

Since (x_n) and (y_n) are positive sequences in $\mathcal{C}$, there exist positive rationals ε_1, ε_2 and $n_1, n_2 \in \mathbb{N}$ such that $x_n > \varepsilon_1$ for all $n \geq n_1$ and $y_m > \varepsilon_2$ for all $m \geq n_2$. Choose $N = \max(n_1, n_2)$. Then for all $n \geq N$, $x_n + y_n > \varepsilon_1 + \varepsilon_2$. This proves that $(x_n + y_n)$ is a positive sequence in $\mathcal{C}$. Proof of other part is similar and we leave it for the reader to complete. □

We denote the set of all positive sequences in $\mathcal{C}/\mathcal{N}$ by $\mathbb{R}^+$.

Definition 16. Let $\mathcal{F}$ be a field. By an order on $\mathcal{F}$ we mean a subset $\mathcal{F}^+$ of $\mathcal{F}$ with the following properties:

1. Any $x \in \mathcal{F}$ lies in exactly one of the sets $\mathcal{F}^+$, $\{0\}$, and $\mathcal{F}^- := -\mathcal{F}^+$.
2. For any x, $y \in \mathcal{F}^+$ their sum $x + y$ and the product xy again lie in $\mathcal{F}^+$.

Theorem 17. $\mathbb{R}$ *is an ordered field with an order* $\mathbb{R}^+$.

Proof. Follows directly from Theorem 2. □

Definition 18. Let $\bar{x}, \bar{y} \in \mathbb{R}$. We say that $\bar{x} > \bar{y}$ if $\bar{x} - \bar{y} \in \mathbb{R}^+$.

Theorem 19. *$\mathbb{R}$ has the Archimedian property, that is, if $\bar{x} \in \mathbb{R}^+$ and $\bar{y} \in \mathbb{R}$, then there exists $n \in \mathbb{N}$ such that $n\bar{x} > \bar{y}$.*

Proof. Since $\bar{x} \in \mathbb{R}^+$, there exists $\varepsilon \in \mathbb{Q}$ and $n_0 \in \mathbb{N}$ such that $x_n > \varepsilon$, for all $n \geq n_0$. Since $y \in \mathcal{C}$, there exists $M \in \mathbb{Q}^+$ such that $|y_n| \leq M$, for all $n \in \mathbb{N}$. It follows from the Archimedian property in $\mathbb{Q}$ that for the above ε and M in $\mathbb{Q}$ there exists N in $\mathbb{N}$ such that $N\varepsilon > M + \varepsilon$. Hence, for all $n \geq n_0$, $Nx_n > N\varepsilon > M + \varepsilon > y_n + \varepsilon$. That is, $Nx_n - y_n > \varepsilon$, for all $n \geq n_0$. Thus $(Nx_n - y_n) \in \mathbb{R}^+$ and hence $N\bar{x} > \bar{y}$. □

Corollary 20. *$\mathbb{N}$ is not bounded in $\mathbb{R}$.* □

Theorem 21. *$\mathbb{R} = \mathcal{C}/\mathcal{N}$ has the l.u.b. property, that is, if S is a non empty subset of $\mathbb{R}$ which is bounded above, then there exists a real number which is the least upper bound for S.*

Proof. Let $S \subseteq \mathbb{R}$ be non empty and bounded above. Let $M \in \mathbb{R}$ be such that $x \leq M$, for all $x \in S$. Without loss of generality we can assume that $M \in \mathbb{Z}$. (Use the above corollary.) Fix $x \in S$. We claim that there exists $m \in \mathbb{Z}$ such that $m \leq x$. For, otherwise $m > x$, for all $m \in \mathbb{Z}$, which implies that $-m < -x$, for all $m \in \mathbb{Z}$. This implies that $\mathbb{N}$ is bounded, which is a contradiction. Hence, $m \leq x \leq M$, for some $m, M \in \mathbb{Z}$. Since S is bounded above by M, if at all the lub exists, it has to lie in $[m, M]$. For each $n \in \mathbb{N}$, consider the set

$$B_n = \{\frac{c}{2^n} \mid m \leq \frac{c}{2^n} \leq M, c \in \mathbb{Z}\}.$$

Note that, since $M = \frac{M \cdot 2^n}{2^n}$, $M \in B_n$. Hence B_n is non empty. Also if $\frac{c}{2^m}$, then $\frac{2c}{2^{n+1}} \in B_{n+1}$. Hence $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$. Since there are only finietly many integers between $m2^n$ and $M2^n$, B_n is finite. Hence there are only finitely many upper bounds for S in B_n. Let a_n be the smallest such upper bound for S in B_n. Since, for $n \geq m$, $B_m \subseteq B_n$, it follows that $a_m \in B_n$ and hence $a_n \leq a_m$ for all $n \geq m$.

Now we claim that for each $n \in \mathbb{N}$, $a_n - \frac{1}{2^n}$ is not an upper bound for S. If $m \leq a_n - \frac{1}{2^n}$, then we are through; since a_n is the least upper bound for S in B_n, $a_n - \frac{1}{2^n} < a_n$ cannot be an upper bound.

Suppose $m > a_n - \frac{1}{2^n}$. Since $m \leq x$, for $x \in S$, $a_n - \frac{1}{2^n} < m \leq x$. Hence $a_n - \frac{1}{2^n}$ cannot be an upper bound for S. Since $a_m - \frac{1}{2^m} \leq x \leq a_n$, it follows that $a_m - a_n < \frac{1}{2^m}$. Hence (a_n) is a Cauchy sequence. Define $\alpha := (a_n) + \mathcal{N}$. First of all we claim that $a_n - \frac{1}{2^n} < \alpha < a_n$, where $a_n = (a_n, a_n, \ldots)$ and $a_n - \frac{1}{2^n} = (a_n - \frac{1}{2^n}, a_n - \frac{1}{2^n}, \ldots)$. Since a_k is a decreasing sequence $a_n \geq a_k$, for all $k \geq n$. Hence $a_n - a_k > 0$, for all $k \geq n$. This shows that $(a_n - \alpha)$ is a positive sequence in $\mathcal{C}$ and hence $\alpha < a_n$. Next $\alpha - (a_n - \frac{1}{2^n}) = (a_1 - a_n + \frac{1}{2^n}, \ldots)$. Since $a_k - \frac{1}{2^k}$ is not an upper bound for S, $a_k > (a_n - \frac{1}{2^n})$, for all k, $a_k - (a_n - \frac{1}{2^n}) > 0$, for all k. Hence $\alpha - (a_n - \frac{1}{2^n})$ is a positive sequence and hence $\alpha > a_n - \frac{1}{2^n}$.

Finally we claim that α is the least upper bound for S. First of all we have to show that α is an upper bound for S. Suppose not. Then there exists $x \in S$ such that $x > \alpha$, hence $x - \alpha > 0$. Using the Archimedian property there exists $N \in \mathbb{N}$ such that $N(x - \alpha) > 1$. Hence $x - \alpha > \frac{1}{N} > \frac{1}{2^N}$. Thus we see that $x > \alpha + \frac{1}{2^N} > a_N - \frac{1}{2^N} + \frac{1}{2^N} = a_N$. This is a contradiction, since a_N is the least upper bound for S in B_N.

Next to show that α is the least upper bound. Suppose not. Then there exists $b \in \mathbb{R}$ such that b is an upper bound for S and $b < \alpha$. By the Archimedian property there exists $N \in \mathbb{N}$

such that $N(\alpha - b) > 1$. Hence $\alpha - b > \frac{1}{N} > \frac{1}{2^N}$. This implies that $\alpha - b > \frac{1}{2^N}$ and hence $\alpha - \frac{1}{2^N} > b$, or, $b < \alpha - \frac{1}{2^N} < a_N - \frac{1}{2^N}$, as $\alpha < a_n$, for all n. But $a_N - \frac{1}{2^N}$ is not an upper bound for S. Hence anything less than this cannot be an upper bound for S. In particular, b is not an upper bound for S, which is a contradiction. This proves that α is the least upper bound for S. □

Remark 22. This is based on a set of notes prepared by my student Ajit Kumar based on my lectures. I thank him for the notes.