# Banach Contraction Principle

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Throughout this section we let $X$ stand for a metric space $(X, d)$ unless specified otherwise.

**Definition 1.** Let $X$ and $Y$ be metric spaces. A map $T: X \to Y$ is said to be a *contraction* if there exists a constant $c$, $0 < c < 1$ such that

$$d(T(x), T(x')) \le c d(x, x'), \qquad \text{for all } x, x' \in X.$$

Notice that any contraction is Lipschitz continuous so that it is uniformly continuous.

**Ex. 2.** Let $f: [a, b] \to [a, b]$ be differentiable and $|f'(x)| \le c$ with $0 < c < 1$. Then $f$ is a contraction of $[a, b]$.

**Theorem 3.** *Let $(X, d)$ be a complete metric space. Assume that $T: X \to X$ is a contraction. Then $f$ has a unique fixed point — a point $x \in X$ such that $f(x) = x$.*

*In fact, if we take any $x_0 \in X$ and let $x_n := T(x_{n-1})$ be defined recursively for $n \ge 1$, then $(x_n)$ converges to an $x \in X$. Furthermore we have*

$$d(T^n x_0, x) \le \frac{c^n}{1-c} d(x_0, Tx_0). \tag{1}$$

*Proof.* For $m < n$ observe that

$$\begin{aligned} d(x_m, x_n) &\le c^m (x_0, T^{n-m}(x_0)) \\ &\le c^m \left[d(x_0, x_1) + \cdots + d(x_{n-m-1}, x_{n-m})\right] \\ &\le c^m d(x_0, x_1) \left[1 + c + \cdots + c^{n-m-1}\right] \\ &\le c^m \frac{d(x_0, x_1)}{1-c}. \end{aligned}$$

Thus $(x_n)$ is Cauchy. let $x := \lim x_n$. Using continuity of $T$ we see that $x = \lim_n T(T^n x) = Tx$. Uniqueness is easy. ∎

We offer another simple proof of the contraction lemma.

**Theorem 4.** *Let $(X, d)$ be a complete metric space. Assume that $T: X \to X$ is a contraction, say, $d(Tx, Ty) \le \lambda d(x, y)$ for a fixed $0 < \lambda < 1$. Then $f$ has a unique fixed point, a point $x \in X$ such that $T(x) = x$.*

*Proof.* It is easy to see by induction that for any $n \in \mathbb{N}$, we have $d(T^n x, T^n y) \le \lambda^n d(x,y)$. Observe that for any $x, y \in X$, we have the following inequalities:

$$\begin{aligned} d(x,y) &\le d(x,Tx) + d(Tx,Ty) + d(Ty,y) \\ d(x,y) &\le d(x,TX) + \lambda d(x,y) + d(y,Ty) \\ (1-\lambda)d(x,y) &\le d(x,Tx) + d(y,Ty) \\ d(x,y) &\le \frac{1}{1-\lambda}\left(d(x,Tx) + d(y,Ty)\right). \end{aligned} \tag{2}$$

Fix $x \in X$. For $m, n \in \mathbb{N}$, substitute $T^m x$ and $T^m x$ for $x$ and $y$ respectively in (2) and get

$$\begin{aligned} d(T^n x, T^m x) &\le \frac{1}{1-\lambda}\left(d(T^n x, T^{n+1}x) + d(T^m x, T^{m+1}x)\right) \\ &\le \frac{1}{1-\lambda}\left(\lambda^n d(x,Tx) + \lambda^m d(x,Tx)\right) \\ &= \frac{d(x,Tx)}{1-\lambda}\left(\lambda^n + \lambda^m\right). \end{aligned} \tag{3}$$

Since $0 < \lambda < 1$, (3) shows that the sequence $(T^n x)$ is Cauchy. Since $X$ is complete, it is convergent, say, to $p \in X$. Since $T^n x \to p$, by continuity, $T(T^n x) \to Tx$. But $T(T^n x) = T^{n+1}x \to p$. By the uniqueness of the limit, we deduce $Tp = p$.

The fixed point $p$ is unique. For if $Tp = p$ and $Tq = q$ with $p \ne q$, then

$$d(p,q) = d(Tp,Tq) < \lambda d(p,q) < d(p,q),$$

a contradiction. □

**Ex. 5.** Let $X$ be complete. Let $T\colon X \to X$ be continuous. Assume that $T^n$ is a contraction for some $n \ge 1$. Then $T$ has a unique fixed point.

**Ex. 6.** Show that the conclusion in Theorem 4 is false if we assume the simpler condition $d(Tx,Ty) < d(x,y)$ for all $x \ne y$. *Hint:* Consider $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x + \frac{1}{x}, & x \ge 1 \\ x - \frac{1}{s-2}, & x < 1. \end{cases}$$

**Ex. 7.** Let $X$ be compact. Let $T\colon X \to X$ be such that $d(T(x),T(y)) < d(x,y)$ if $x \ne y$. Then $T$ has a unique fixed point.

**Ex. 8.** Find a compact metric space $X$, $T\colon X \to X$ be such that $d(T(x),T(y)) \le d(x,y)$ for all $x, y \in X$ while $T$ has no fixed point. Find also $X$ and $T$ satisfying this condition such that $T$ has more then one fixed point.

**Ex. 9.** Let $f\colon [a,b] \to \mathbb{R}$ be differentiable with $f(a) < 0 < f(b)$ and $0 < m_1 \le f'(x) \le m_2$ for $x \in [a,b]$. Then $f$ has a unique zero in $[a,b]$. *Hint:* Consider $g(x) := x - \frac{1}{m}f(x)$. Then $x$ is a zero of $f$ iff $x$ is a fixed point of $g$. Choose $m$ appropriately so that $g$ is a contraction of $[a,b]$ into itself. If $m > m_2$ then take $c := 1 - \frac{m_1}{m}$. If $0 < m_2 < 2m_1$ then let $m \in (m_2/2, m_1]$ and $c := \frac{m_2}{m} - 1$.

**Ex. 10.** Let $A$ be an $n \times n$-matrix with real entries, $x, y \in \mathbb{R}^n$. Consider $Ax = y$ or

$$\sum_{j=1}^{n} a_j^i x^j = y^i, \qquad 1 \le i \le n.$$

Then if either $\lambda_\infty := \max_i \sum_{j=1}^n |\delta_j^i - a_j^i| < 1$ or if $\lambda_1 := \max_j \sum_{i=1}^n |\delta_j^i - a_j^i| < 1$, the equation $Ax = y$ has a unique solution. *Hint:* Consider $T := T_y$ where $Tx := (I - A)x + y$. If $x_r := (x_r^1, \dots, x_r^n)$ for $r = 1, 2$ and $y_r := T(x_r)$ then $d_\infty(y_1, y_2) \le \lambda_\infty d_\infty(x_1, x_2)$ etc.

## ODE and Integral Equations

We shall define the notion of integrals of vector valued functions of a real variable. We wish to assign a meaning for the integral of a vector valued function in such a way that the "natural" results such as the fundamental theorem of calculus and the inequality that $\left\| \int_a^b f(t)\,dt \right\| \le \int_a^b \| f(t) \| \, dt$ is true.

**Definition 11.** Let $f : [a, b] \to \mathbb{R}^n$ be a continuous function. We write $f = (f_1, \dots, f_n)$ and set

$$\int_a^b f(t)\,dt := (\int_a^b f_1(t)\,dt, \dots, \int_a^b f_n(t)\,dt).$$

**Ex. 12.** Keep the notation in the definition. Show that the map $w \mapsto \int_a^b \langle f(t), w \rangle \, dt$ is a linear functional on $\mathbb{R}^n$ and hence by Riesz there exists a unique vector $v$ such that this map is $w \mapsto \langle w, v \rangle$. Show that $v = \int_a^b f(t)\,dt$. Hence $\int_a^b f(t)\,dt$ is that vector $v \in \mathbb{R}^n$ such that

$$\left\langle \int_a^b f(t)\,dt, w \right\rangle = \int_a^b \langle f(t), w \rangle \, dt, \qquad \text{for all } w \in \mathbb{R}^n. \tag{4}$$

**Ex. 13.** Prove the following versions of the fundamental theorems of calculus:

1. Let $f : [a, b] \to \mathbb{R}^n$ be continuous. Define $g(t) := \int_a^t f(s)\,ds$. Then $g$ is continuously differentiable and $g'(t) = f(t)$.
2. Let $g : [a, b] \to \mathbb{R}^n$ be continuously differentiable. Then $\int_a^b g'(t)\,dt = g(b) - g(a)$.

**Ex. 14.** With appropriate hypothesis, show that $\left\| \int_a^b f(t)\,dt \right\| \le \int_a^b \| f(t) \| \, dt$. *Hint:* In Eq. nint take $w := \int_a^b f(t)\,dt$ and use Cauchy-Schwarz.

Consider the vector differential equation

$$x'(t) = f(t, x(t)), \qquad \text{with the initial condition } x(0) = x_0. \tag{5}$$

This is equivalent to the system of $n$ first order DE's in $n$ unknowns with the initial conditions (IC's):

$$x_i'(t) = f^i(t, x_1(t), \dots, x_n(t)), \qquad x_i(0) = (x_0)_i, 1 \le i \le n.$$

**Ex. 15.** The above DE with IC is equivalent to the following integral equations

$$x(t) = x_0 + \int_0^t f(s, x(s))\,ds.$$

**Theorem 16.** *Let $f\colon [a,b]\times\mathbb{R}^n \to \mathbb{R}^n$ be continuous. Assume that there exists a constant $L$ such that*

$$\| f(t,x) - f(t,y)\| \le L\,\|x-y\|\,, \qquad x,y\in\mathbb{R}^n, t\in[a,b].$$

*Then the integral equation $x(t) = x_0 + \int_0^t f(s,x(s))\,ds$ has a unique solution on $[a,b]$.*

*Proof.* Let $X := C([a,b],\mathbb{R}^n)$ be the vector space on which we define a norm $\|x\|_\alpha := \sup\{e^{-\alpha(t-a)}\|x(t)\| : t\in[a,b]\}$. It is easily seen that there exist constants $C_i$ such that $C_1\|x\| \le \|x\|_\alpha \le C_2\|x\|$ where $\|x\| := \sup\{\|x(t)\| : t\in[a,b]\}$ and that $(X,\|\cdot\|_\alpha)$ is continuous iff $(X,\|\cdot\|)$ is complete. Let $T\colon X\to X$ be given by $Tx(t) := x_0 + \int_a^t f(s,x(s))\,ds$. Show that for $\alpha$ very large, $T$ is a contraction on $(X,\|\cdot\|_\alpha)$. □

**Ex. 17.** Let the notation be as in Theorem 16. Then the differential equation $x'(t) = f(t,x(t))$ with the initial condition $x(a) = x_0$ has a unique solution.

**Ex. 18.** Let the notation be as in Theorem 16. If we use the norm $\|\cdot\|$ on $X$ we can solve Eq. 5 on $(-\varepsilon,\varepsilon)$ for some $\varepsilon>0$.

**Definition 19.** A nonlinear Volterra integral equation of the 2nd kind is of the form

$$x(t) = g(t) + \lambda\int_0^t f(t,s,x(s))\,ds. \tag{6}$$

**Theorem 20.** *Let $g\colon\mathbb{R}\to\mathbb{R}^n$ be Lipschitz: $\|g(s)-g(t)\|\le L|s-t|$ in an interval $J$ around 0. Assume that $g(0)=0$. Let $f\colon\mathbb{R}\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ be continuous and satisfy a Lipschitz condition in the last $n$ variables uniformly in the first two variables:*

$$\| f(t,s,v) - f(t,s,w)\| \le M\,\|v-w\|$$

*in a neighbourhood $U$ of $(0,0,g(0))$. Then for every $\varepsilon>0$ such that $I := [-\varepsilon,\varepsilon]\subset J$ and $V := I\times I\times B(0,r)\subset U$ the equation Eq. 6 has a unique solution on $I$ for each $\lambda\in\mathbb{R}$.*

*Proof.* Let $X := \{x\colon I\to B[0,r] : x \text{ is continuous}\}$. Consider the following metric on $X$:

$$d(x,y)_\alpha := \sup\{\|x(t)-y(t)\|\,e^{-\alpha|t|} : t\in I\},$$

for $\alpha>0$. One shows that this metric is uniformly equivalent to $d_0$ in the sense that there exist constants $C_1$ and $C_2$ such that $C_1 d_0(x,y)\le d_\alpha(x,y)\le C_2 d_0(x,y)$. It is easily seen that $(X,d_0)$ is complete and hence so is $(X,d_\alpha)$. Let $T$ be defined by $Tx(t) := g(t) + \lambda\int_0^t f(t,s,x(s))\,ds$. We choose $\alpha$ sufficiently large so that i) $T$ maps $X$ into itself and ii) $T$ is a contraction. □

**Corollary 21.** *Let $J$ be an interval around $0\in\mathbb{R}$ and $U$ be the open ball $B(x_0,r)\subset\mathbb{R}^n$. Let $f\colon J\times\mathbb{R}^n\to\mathbb{R}^n$ be continuous and satisfy the Lipschitz condition*

$$\| f(t,v) - f(t,w)\| \le L\,\|v-w\|$$

*in $J\times U$. Then for each $a>0$ such that*

$$S := \{(t,v) : |t|\le a, \|v-x_0\|\le b\}\subset J\times U,$$

*there exists a unique solution of Eq. 5 on $[-a,a]$.* □

In many problems (such as dependence of the solutions of ODE on the initial data) one wants to know the dependence of the fixed points on the parameters of a parameterized family of maps. The following result is in this direction.

**Theorem 22.** *Let $X$ be a complete metric space and $\Lambda$ be a metric space. Let $c$, $0 < c < 1$ be fixed. Let $F\colon \Lambda \times X \to X$ be continuous. We let $f_\lambda(x) := F(\lambda, x)$. Assume that $d(f_\lambda(x), f_\lambda(y)) \leq c d(x, y)$ for a all $x, y \in X$ and $\lambda \in \Lambda$. Then*

1. *For each $\lambda \in \Lambda$ there exists a unique fixed point $x_\lambda$ of $f_\lambda$.*
2. *The map $\lambda \to x_\lambda$ is continuous.*

*Proof.* To prove 2), in Eq. 1 we take $n = 0$, $T = f_\mu$, $x = x_\mu$ and $x_0 = x_\lambda$. Then

$$d(x_\lambda, x_\mu) \leq \frac{1}{1-c} d(x_\lambda, f_\mu(x_\lambda)). \tag{7}$$

As $\mu \to \lambda$, $f_\mu(x_\lambda) = F(\mu, x_\lambda) \to F(\lambda, x_\lambda) = f_\lambda(x_\lambda) = x_\lambda$. Hence the RHS of Eq. 7 goes to 0. □

**Ex. 23.** Formulate and prove a theorem to the effect that the solution of initial value problem Eq. ODE depends continuously on the initial data $x_0$.

**Remark 24.** See also the section on ODE for proofs of Corollary 21 and this exercise.

**Definition 25.** An integral equation of the form

$$x(t) = g(t) + \lambda \int_a^b K(t, s, )x(s)\, ds. \tag{8}$$

is called the Fredholm integral equation of the second kind. $K$ is called the kernel.

**Theorem 26.** *Let $g\colon I := [a, b] \to \mathbb{R}^n$ be continuous. Let $K\colon I \times I \to \mathbb{R}$ be continuous. Then the integral equation Eq. 8 has a unique solution $x$ on the interval $[a, b]$ for each $\lambda$ such that $|\lambda| < \frac{1}{M(b-a)}$, where $M$ is a bound for $K$ on $I^2$.*

*Proof.* Let $X := \{x\colon [a, b] \to \mathbb{R}^n : x \text{ is continuous}\}$ with the sup norm. Define $T\colon X \to X$ by $Tx(t) := g(t) + \lambda \int_a^b K(t, s)x(s)\, ds$. Then $T$ is a contraction. □