Correspondence Theorem for Groups

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Let G be a group and H a normal subgroup. Consider the quotient group G/H of left cosets. Let $\pi\colon G \to G/H$ be the quotient map $\pi(g) := gH$. We prove the following theorem and give its applications.

Theorem 1. *Let G be a group and H a normal subgroup. Consider the quotient group G/H. Let $\mathcal{L}$ be the set of all subgroups of G/H and $\mathcal{K}$ be the subgroups of G containing H. Then the map*

$$\varphi\colon \mathcal{L} \to \mathcal{K}, \text{ given by } L \mapsto \pi^{-1}(L)$$

is a bijection.

Moreover, if L is a normal subgroup of G/K, then $K := \pi^{-1}(L)$ is a normal subgroup of G.

Finally, there is a bijection of $\pi^{-1}(L)$ with $L \times H$ for any $L \in \mathcal{L}$.

Proof. Let $L \in \mathcal{L}$ and let $K := \pi^{-1}(L)$. We show that K is a subgroup of G. Let $x, y \in K$. That is, $x, y \in \pi^{-1}(L)$. Hence $\pi(x), \pi(y) \in L$. Hence $\pi(xy) = \pi(x)\pi(y) \in L$, since L is a subgroup. This means that $xy \in \pi^{-1}(L) = K$. Similarly, if $x \in K$, $\pi(x) \in L$ so that $\pi(x)^{-1} \in L$. Since π is a group homomorphism, $\pi(x)^{-1} = \pi(x^{-1})$. Thus, $\pi(x^{-1}) \in L$ or $x^{-1} \in K$. Thus we have established that K is a subgroup of G. Also, if $x \in H$, $\pi(x) = eH$ and hence $\pi^{-1}(e) = H \subset K$. Thus $K \in \mathcal{K}$.

We now show that if L is normal in G/H, then $K := \pi^{-1}(L)$ is normal in G. Let $x \in K$ and $g \in G$. We have

$$\pi(gxg^{-1}) = \pi(g)\pi(x)\pi(g^{-1}) = \pi(g) \cdot \pi(x) \cdot \pi(g)^{-1}.$$

Since $\pi(x) \in L$ and L is normal in G/H, it follows that$\pi(g)\pi(x)\pi(g)^{-1} \in L$. We therefore conclude that $\pi(gxg^{-1}) \in L$. This means that $gxg^{-1} \in K$. Hence K is normal in G.

We now show that that map φ is a one-one. Let $L_1 \neq L_2 \in \mathcal{L}$. Hence there exists an element y in one of them and not in the other. Without loss of generality, assume that $y \in L_2 \setminus L_1$. Since π is onto, there exists $x \in G$ such that $\pi(x) = y$. We claim that $x \in K_2 \setminus K_1$. Since $\pi(x) = y \in L_2$, we see that $x \in K_2$. If $x \in K_1 = \pi^{-1}(L_1)$, it follows that $y = \pi(x) \in L_1$, a contradiction. Hence we conclude that $\varphi\colon \mathcal{L} \to \mathcal{K}$ is one-one.

We now claim that that φ is onto. Let $K \in \mathcal{K}$. Let $L := \pi(K)$. One easily shows that L is a subgroup of G/H. We claim that $K = \pi^{-1}(L)$ so that $\varphi(L) = K$. First of all, observe that $K \subset \pi^{-1}(L)$. For, if $x \in K$, then $\pi(x) \in \pi(K) = L$. Hence $x \in \pi^{-1}(L)$. We now show that $\pi^{-1}(L) \subset K$. Let $x \in \pi^{-1}(L)$. Thus, $\pi(x) = L$. But since $\pi(x) \in \pi(K)$, there exists $g \in K$ such that $\pi(x) = gH$. Recall that $\pi(x) = xH$, So we have $xH = gH$ or $g^{-1}x \in H$. Since $H \subset K$, we see that $g^{-1}x \in K$. By choice $g \in K$ so that $x = g(g^{-1}x) \in K$. Thus $x \in K$. We have therefore shown that $\pi^{-1}(\pi(K)) = K$. This establishes that φ is onto. We have thus proved that $\varphi\colon \mathcal{L} \to \mathcal{K}$ is a bijection.

Let us now prove the last part. Let $L := \{g_i H : i \in I\}$. We claim that $\pi^{-1}(L) = \{g_i h : i \in I, h \in H\}$. For, let $x \in \pi^{-1}(L)$. Hence $\pi(x) \in L$. Thus we can find a $j \in I$ such that $\pi(x) = g_j H$. But by the very definition of π, we have $\pi(x) = xH$. It follows that $xH = g_j H$. We deduce that $g_j^{-1}x \in H$, say, $g_j^{-1}x = h$. Thus $x = g_j h$. Hence the claim is proved.

We now claim that all the elements in the set $\{g_i h : i \in I, h \in H\}$ are distinct. Let $g_i h = g_j h_1$. We get $g_j^{-1} g_i = h_1 h^{-1} \in H$. It follows that $g_i H = g_j H$ and hence $g_i = g_j$. Since $g_i h = g_j h_1 = g_i h_1$, we find that $h = h_1$. Thus the claim is proved. The map $L \times H \to \pi^{-1}(L)$ given by $(g_i H, h) \mapsto g_i h$ is a bijection. $\square$

A special case of the last part: if G is finite, we have $|\pi^{-1}(L)| = |L||H|$.

The result in the last part of the theorem is reminiscent of the following results from linear algebra.

Let $T\colon V \to W$ be an onto linear map. Fix $y \in W$. Since T is onto, there is $x \in V$ such that $Tx = y$. We claim that the set of all solutions of $Tx = y$ is the set $x + \ker T$. For, if $z \in \ker T$, then $T(x + z) = Tz + Tz = y + 0 = y$. Hence $x + \ker T \subset T^{-1}(y)$. Conversely, if $v \in V$ satisfies $Tv = y$, we than have $Tx - Tv = 0$ so that $T(x - v) = 0$. Thus, $x - v \in \ker T$, say $x - v = z \in \ker T$. We have $v = x - z \in x + \ker T$.

Let us apply the last theorem.

Theorem 2. *Let G be a group of order p^n. Then for each r with $0 \le r \le n$ there exists a subgroup of order p^r.*

Proof. The proof is by induction on n. When $n = 1$, we have $|G| = p$ so that $r = 0$ or $r = 1$. The subgroups are accordingly the trivial and full groups.

Assume that the result holds true for groups of order p^{n-1} when $n \ge 2$.

Let G be a group of order p^n. Since G is a p-group, its center $Z(G)$ is not trivial so that $|Z(g)| = p^r$ with $r > 0$. By Cauchy's theorem, there exists an element $a \in Z(G)$ of order p. Since the cyclic group $< a > \subset Z(G)$, the subgroup $< a >$ is normal in G. (For, if $g \in G$, $ga^i g^{-1} = a^i g g^{-1} = a_i$, since a_i commutes with all elements of G.)

The quotient group $G/ < a >$ is of order p^{n-1}. Let $0 \le r \le n - 1$. Then by indction hypothesis, there exists a subgroup $L \le G/ < a >$ whose order is p^r. By the last result, $K := \pi^{-1}(L)$ is a subgroup of G of order $|L|| < a > | = p^{r+1}$. Thus G has subgroups of order p^r, $1 \le r \le n$. The result is proved. $\square$