Covering Spaces

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In the following, we assume that our spaces are Hausdorff, connected topological manifolds. The main purpose of this article to establish the fact that path lifting local homeomorphisms and covering maps are one and the same. See Thm. 11.

Let X, X' be two n-dimensional manifolds. Let $\pi : X' \to X$ be such that for every $x' \in X$, there exists an open set $U' \in X'$, with $x' \in U'$ such that $\pi(U') = U$ and $\pi\,|_{U'}\colon U' \to U$ is a homeomorphism. (Such maps are called local homeomorphisms.)

Let Y be any topological space and $c : Y \to X$ be a continuous map. A *lift* c' of c (to X' or with respect to π) is a continuous map $c' : Y \to X'$ such that $\pi \circ c' = c$. If a lift c' exists, we say that c can be lifted.

Lemma 1. *Let $\pi : X' \to X$ be a local homeomorphism and Y a connected Hausdorff space. Let $f : Y \to X$ be a continuous map. Let f_i be a lift of f, $i = 1, 2$. Assume that there exists $y_0 \in Y$ such that $f_1(y_0) = f_2(y_0)$. Then, we have $f_1 = f_2$ on Y.*

Proof. Let $E := \{y \in Y \mid f_1(y) = f_2(y)\}$. Then $y_0 \in E$ and hence E is non-empty. Since Y is Hausdorff and f_i are continuous, E is closed.

Since Y is connected, it is enough to prove that E is open.

Let $y \in E$ and let $x' := f_1(y) = f_2(y)$. By hypothesis, there exists an open neighborhood U' of x' in X' such that $U := \pi(U')$ is open and $\pi\,|_{U'}\colon U' \to U$ is a homeomorphism. Since f_i are continuous from Y to X', there exists a neighborhood V of y such that $f_i(V) \subseteq U'$. Now, if $z \in V$, then $\pi \circ f_i(z) = f(z) = \pi \circ f_2(z)$. Since π is one-one on U', this implies $f_1(z) = f_2(z)$, for any $z \in V$, a neighborhood of y. That is, $y \in V \subseteq E$ and hence E is open. □

Definition 2. Let X, X' be manifolds. Let $\pi : X' \to X$ be a continuous map. π is said to be a *covering map* (or X' is a *covering of* X) if for every $x \in X$, we can find a neighborhood U of x such that $\pi^{-1}(U)$ is a disjoint union $\pi^{-1}(U) := \cup U'_\alpha$ where U'_α is open in X' and $\pi : U'_\alpha \to U$ is a surjective homeomorphism.

U as above is called an *admissible neighborhood* of x; or U is *evenly covered* by π. (Note that if U is connected, the U'_α's are nothing other than connected components of $\pi^{-1}(U)$.)

Remark 3. A covering map is obviously a local homeomorphism.

If U is an admissible neighborhood of x and $V \subseteq U$ is a neighborhood of x, then V is an admissible neighborhood of x.

The characterizing property of a covering map is the existence of lifts of curves $c : I \to X$. In the proposition below, we shall prove this one way.

Proposition 4. *Let $\pi : X' \to X$ be a covering map. Let $a' \in X'$ and $a := \pi(a')$. Let $I := [0,1]$. Let $c : I \to X$ be a curve with $c(0) = a$. Then there exists a unique lift $c' : I \to X$ with $c'(0) = a'$.*

Proof. Let U_x denote an admissible neighborhood of x, for $x \in X$. Since $c(I)$ is compact, there exists a finite family $U_i := U_{x_i}$ of admissible neighborhoods and a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of I such that $c([t_i, t_{i+1}]) \subseteq U_i$, $i = 0, 1, \ldots, n-1$. Let V_0 be the unique connected component of $\pi^{-1}(U_i)$ such that $a' \in V_0$. Since $\pi\mid_{V_0} : V_0 \to U_i$ is a homeomorphism. We can define $c' : [t_0, t_1] \to V_0$ by pulling c via $(\pi\mid_{V_0})^{-1}$

$$c' := (\pi\mid_{V_0})^{-1} \circ c, \qquad \text{on } [t_0, t_1].$$

We now proceed by an obvious induction. Assume that c' is defined on $[0 = t_0, t_i]$, $i \geq 1$. Let V_i be the connected component of $\pi^{-1}(U_1)$ containing $c'(t_i)$. We define $c' : [t_i, t_{i+1}] \to V_i$ using the homeomorphism $\pi\mid_{V_i} : V_i \to U_i$ as above. Then c' is defined continuously on $[t_0, t_{i+1}]$. Uniqueness follows from Lemma 1. □

Definition 5. Let $c_i : [0,1] \to X$, $i = 0, 1$ be two curves. We say that c_0 is *homotopic* to c_1 (or c_0 and c_2 are homotopic) if there exists a continuous map $H : I \times I \to X$ such that $H(s, 0) = c_0$ and $H(s, 1) = c_1$. We let $c_t(s) := H(s,t)$. H is called a *homotopy* between c_0 and c_1.

Let $c_0(0) = c_1(0) = a$. F is said to *fix* the end point a if

$$\begin{aligned} c_t(0) &= c_0(0) = a \text{ for all } t \in I \text{ and} \\ c_t(1) &= c_0(1) = b \text{ for all } t \in I. \end{aligned}$$

Theorem 6 (Monodromy Theorem)**.** *Let X, X' be manifolds. Let $\pi : X' \to X$ be a local homeomorphism. Let $a = \pi(a')$, $b \in X$. Let c_0, c_1 be curves in X which start from a and end at b. Let $H := c_t$ be a homotopy between c_0 and c_1 leaving the end points fixed.*

Let c_0', c_1' be lifts of c_0 and c_1 having a' as their initial point.

Then they have the same endpoint and there exists a unique homotopy $H' : I \times I \to X'$ of c_0' to c_1' such that H' is a lift of H.

Furthermore, H' fixes the endpoints also. (In particular, $c_t'(1) = c_0'(1) = c_1'(1)$, for all $t \in [0,1]$.)

Proof. By Prop. 4, each path $c_t : I \to X$ has a unique lift $c_t' : I \to X'$, with $c_t'(0) = a'$. We let $H'(s,t) := c_t'(s)$. Thus $H' : I \times I \to X'$ with $\pi \circ H' = H$. But continuity of H' is to be established.

We mimic the idea of the proof of Prop. 4. Fix $t_0 \in I$. We choose a partition of the line $I \times \{0\} \subseteq I \times I$ by $0 = s_0 < s_1 < \cdots < s_n = 1$ such that each interval $\{[s, t_0] \mid s_i \leq s \leq s_{i+1}\}$ is carried by $H \mid_{I \times \{t_0\}} = c_{t_0}$ into an admissible neighborhood V_i of X. Let V_i be the connected component of $\pi^{-1}(U_i)$ containing $\widetilde{c}_{t_0}(s_i)$. Note that, we have, $\widetilde{c}_{t_0}(s) = (\pi \mid_{V_i})^{-1}(c_{t_0}(s))$ for $s_i \leq s_{i+1}$. By continuity of H, there exists $\varepsilon > 0$ such that if Q_i is the cube $Q_i := \{(s,t) \mid s_i \leq s \leq s_{i+1}, t_0 - \partial \leq t \leq t + \partial\}$, then $H(Q) \subseteq U_i$. Hence $\widetilde{H}(s,t) = \widetilde{c}_t(s) = (\pi \mid_{V_i})^{-1}(H(s,t))$ on Q_i. This implies that $\widetilde{H}$ is continuous on Q_i.

This holds true for all i, $0 \leq i \leq n-1$ and hence $\widetilde{H}$ is continuous in a δ-neighborhood of the line $I \times \{0\}$, viz., $\{(s,t) \mid s \in I, |t - t_0| < \partial\}$. Since t_0 is arbitrary, $\widetilde{H}$ is continuous on $I \times I$.

Since $\widetilde{H}$ is continuous, the map $\widetilde{\gamma} : t \mapsto \widetilde{c}_t(1)$ is continuous. $\widetilde{\gamma}$ is clearly a lift of the constant map $\gamma : t \mapsto c_t(1) = b$. Hence by Lemma 1, $\widetilde{\gamma}$ is also the constant map, viz., $\widetilde{\gamma}(t) = \widetilde{c}_t(1) = \widetilde{c}_0(1) = b$. □

Definition 7. Let X be a path connected Hausdorff space. Let $a \in X$, c a *loop* at a, i.e., c is a curve such that $c(0) = a = c(1)$. We say that c is *homotopic* to a constant (loop) if there exists a homotopy $\{c_t\}$ such that $c_0 = c$ and $c_1(t) = a$, for all t, and $\{c_t\}$ fixes the end points. (Thus, c_t are loops.)

Definition 8. A path connected Hausdorff space is said to be *simply connected* if every loop in it is homotopic to a constant.

Definition 9. Let $\pi : X' \to X$ be a local homeomorphism. We say that π has the *curve lifting property* if the following holds: If $a' \in X'$, $c : I \to X$ is a curve with $c(0) = \pi(a') = a$, then there is a lift $c' : I \to X'$ of c with $c'(0) = a'$.

Proposition 10. *Let $\pi : X' \to X$ have curve lifting property. Assume X, X' are connected manifolds and that X is simply connected. Then π is homeomorphism.*

Proof. We first prove that π is surjective. Let $x \in X$. Fix $a \in X$, $a' \in X'$ with $\pi(a') = a$. Let c be a curve joining a and x. If c' is a lift of c, then $\pi(c'(1)) = x$.

We next prove that π is injective. Let $y_1, y_2 \in X'$ with $\pi(y_1) = x = \pi(y_2)$. Let c' be a curve joining y_1 and y_2 in X'. Let $c := \pi \circ c'$. Then c is a loop at x in X. Since X is simply connected, c is homotopic to the constant loop at x, via a homotopy $\{c_t\}$ leaving x, the endpoint(s) fixed. Let c'_t be the lift of c_t with $c'_t(0) = c'(0) = y_1$. (Existence of c'_t follows from the curve lifting property of π and uniqueness from Lemma 1.) By Monodromy theorem, $t \mapsto c'_t(1)$ is a constant, viz., $c'(1) = y_2$.

Since c_1 is the constant loop at x, by Lemma 1, c'_1 must be a constant loop. But $c'_1(0) = y_1$ so that $c'_1(s) = y_1$ for all s. Thus $y_2 = c'(1) = c'_0(1) = c'_t(1) = c'_1(1) = y_1$. Hence π is injective.

Since π is a local homeomorphism, it is an open map. Thus π is bijective, open, continuous. □

We now prove the converse of Prop. 4.

Theorem 11. *Let $\pi : X' \to X$ be a local homeomorphism of connected manifolds. Then π is a covering map iff π has the curve lifting property.*

Proof. To prove the 'if' part. Let $a \in X$. Since X is a manifold, there exists a simply connected neighborhood U of a in X. Let $\pi^{-1}(U) = \cup U'_\alpha$ be the decomposition of connected components. Since π has the curve lifting property, $\pi\,|_{U'_\alpha}\colon U'_\alpha \to U$ has the same property. For, if $a' \in U'_\alpha$, $c : I \to U$ is a curve with $c(0) = \pi(a')$ and c' is a lift of c, then $c'(I) \subseteq U'_\alpha$, since $c'(I)$ is a connected set containing a'.

Prop. 10 now implies that $\pi\,|_{U'_\alpha}\colon U'_\alpha \to U$ is a homeomorphism for all α. □