Problems in Algebraic Topology

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Ex. 1. Given a convex subset C of $\mathbb{R}^n$ with nonempty interior there is a homeomorphism $f\colon C \to \mathbb{D}^n$ such that $f(\partial C) = \mathbb{S}^{n-1}$.

Ex. 2. Given x, $y \in \mathbb{D}^n$ show that there exists a homeomorphism $f\colon \mathbb{D}^n \to \mathbb{D}^n$ such that $f(x) = y$ and f maps $\mathbb{S}^{n-1}$ to $\mathbb{S}^{n-1}$.

Ex. 3. Show that any connected manifold X is homogeneous—given x, $y \in X$ there exists a homeomorphism $\varphi\colon X \to X$ such that $\varphi(x) = y$. *Hint:* The orbits under the group of homeomorphisms are open.

Ex. 4. Let X be a path connected space, U, V open simply connected subsets of X. Assume that $X = U \cup V$ and $U \cap V$ is path connected. Then X is simply connected.

Ex. 5. Apply the last result to show that $\mathbb{S}^n$ ($n \geq 2$) is simply connected.

Ex. 6. The unit sphere is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.

Ex. 7. Show that $\mathbb{S}^n$ is a deformation retract of $\mathbb{D}^{n+1} \setminus \{0\}$.

Ex. 8. Show that the equator $\mathbb{S}^{n-1}$ is a deformation retract of $\mathbb{S}^n \setminus \{\pm e_{n+1}\}$. Hence $\mathbb{S}^n$ minus 2 points and $\mathbb{S}^{n-1}$ have the same homotopy type.

Ex. 9. Let K be a compact subset of $\mathbb{R}^n$. Show that $\mathbb{R}^n \setminus K$ has exactly one unbounded component.

Ex. 10. $f\colon S^1 \to X$ is null-homotopic iff f can be extended to a map $\mathbb{D}^2 \to X$.

Ex. 11. This is a generalisation of the last exercise. Let $f\colon \mathbb{S}^n \to X$ be a continuous map. The following are equivalent:
i) f is null homotopic.
ii) f can be extended to a continuous map $\mathbb{D}^{n+1} \to X$.
iii) If $x_0 \in \mathbb{S}^n$ and $c = f(x_0)$ is the constant map then there is a homotopy $H\colon f \simeq c$ with $F(x_0, t) = f(x_0)$ for all $t \in I$.

Ex. 12. Given X let CX denote the quotient space $X \times I/(X \times \{1\})$. Then CX is called the cone over X and the point $[(x,t)]$ is the vertex of CX. Show that CX is always contractible.

Ex. 13. Every space can be embedded in a contractible space.

Ex. 14. Exhibit $\mathbb{D}^{n+1}$ as a cover over $\mathbb{S}^n$ with vertex 0.

As a rule all spaces are path connected, locally path connected and hausdorff. One may even assume that they are manifolds.

Ex. 15. Show that the following are covering maps:
i) $\pi \colon \mathbb{R} \to \mathbb{R}_+$ given by $\pi(x) = e^x$.
ii) $\pi \colon \mathbb{R} \to S^1$ given by $\pi(x) = e^{2\pi x}$.
iii) $\pi \colon \mathbb{S}^n \to \mathbb{P}^n$, the quotient map.
iv) $\pi_n S^1 \to S^1$ given by $\pi_n(z) = z^n$, $n \in \mathbb{N}$.
v) $\pi_1 \times \pi_2 X_1 \times X_2 \to B_1 \times B_2$ where $\pi_i \colon X_i \to B_i$ is a covering map .
vi) the polar coordinate map $p \colon \{(r,\theta) \in \mathbb{R}^2 : r > 0\} \to \mathbb{R}^2 \setminus \{0\}$ given by $p(r,\theta) = (r\cos\theta, r\sin\theta)$.
vii) $z \mapsto e^z$ from C to $\mathbb{C}^*$.
viii) $z \mapsto z^n$ from C^* to $\mathbb{C}^*$.

Ex. 16. Fix $a, b, cd \in \mathbb{Z}$ with $m := ad - bc \neq 0$. Consider the map $f \colon S^1 \times S^1 \to S^1 \times S^1$ given by $(z, w) \mapsto (z^1 w^b, z^c w^d)$. Then f is a $|m|$ sheeted covering.

Ex. 17. Find three distinct coverings of the torus.

Ex. 18. Prove that covering maps are local homeomorphisms. The converse is not true even if we assume that the map is surjective. *Hint:* Look at the restriction to $(0, \infty)$ of the covering of S^1.

Ex. 19. Any covering map is open.

Ex. 20. Let $p \colon E \to B$ and $p' \colon E' \to B$ be covering maps. Let $f \colon E \to E'$ be continuous such that $p' \circ f = p$. Then f is a covering.

Ex. 21. Let X, Y be connected and path connected. Assume that X is compact and Y is hausdorff. Let f be a local homeomorphism. Then f is onto and is a covering.

Ex. 22. Let G be a group of homeomorphisms of X. We say G acts on X *properly discontinuously* if there is a G-action on the set X such that $x \mapsto g \cdot x$ is a homeomorphism for each $g \in G$ and such that every $x \in X$ has a neighbourhood U such that $gU \cap g'U = \emptyset$ for $g \neq g' \in G$. Assume that G acts properly discontinuously on X. Then $p \colon X \to X/G$ is a covering.

Ex. 23. Show that the following actions are properly discontinuous:
i) $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ via translation: $z \cdot x = z + x$, $z \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$. The quotient space is homeomorphic to $\mathbb{T}^n = S^1 \times \cdots \times S^1$ (n-times).
ii) Let G be the group generated by the homeomorphisms $(x, y) \mapsto (x+1, y)$ and $(x, y) \mapsto (x, y) \mapsto (-x, y+1)$ on $\mathbb{R}^2$. The quotient is a Klein bottle.
iii) Let G be the group generated by the homeomorphism $(x, y) \mapsto (x+1, -y)$ on $\mathbb{R}^2$. The quotient is a Mobius band.
iv) Let $X = \mathbb{R}^n \setminus \{0\}$. Fix $r \in \mathbb{R} \setminus \{0, \pm 1\}$. Let $G = \mathbb{Z}$. Define a G-action on X by $g \cdot x = r^g x$. The quotient here is homeomorphic to $S^1 \times S^{n-1}$.

Ex. 24. If a finite group G acts on a Hausdorff space X such that $(gx = x) \Rightarrow (g = e)$ then G acts properly discontinuously.

Ex. 25. Lens Spaces. Let p and q be relatively prime integers. Consider S^3 as a subset of $\mathbb{C} \times \mathbb{C}$ in the natural way. The homeomorphism $\varphi \S^3 \to \S^3$ given by $(z, w) \mapsto (e^{2\pi i/p}, e^{2\pi q/p})$. This generates a cyclic group G of order p. The quotient S^3/G is called a Lens space and denoted by $L(p, q)$. The quotient map $S^3 \to L(p, q)$ is a covering.

Ex. 26. Can path lifting lemma be proved for surjetive local homeomorphisms?

Ex. 27. Let $p \colon E \to B$ be a covering map. Show that $\pi_* \colon \pi_1(E, e_0) \to \pi_1(B, b_0)$ is one-one.

Ex. 28. Let $p \colon E \to B$ be a covering map. Then $\#\pi^{-1}(x) = \#\pi^{-1}(y)$ for $x, y \in B$.

Ex. 29. Let X and Y be locally compact along wityh our usual hypothesis. Let $f \colon X \to Y$ be a local homeomorphism. Assume further that f is proper. Then f is a covering map.

Ex. 30. Let $f \colon \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map which is proper. Assume that $Df(x)$ is nonsingular for all $x \in \mathbb{R}^n$. Then f is a diffeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$.

Ex. 31. The topologists' sine curve is the space $X \subset \mathbb{R}^2$ given by

$$X := \{(x, \sin 1/x) : 0 < x \leq 1\} \cup \{(0, y) : -1 \leq y \leq 1\}.$$

Compute $H_*(X)$.

Ex. 32. If the inclusion $j \colon A \hookrightarrow X$ is a homotopy equivalence then $H_p(X, A) = 0$.

Ex. 33. Let $X = A \cup B$ be a disconnection. Then $H_n(X) = H_n(A) \oplus H_n(B)$, for all $n \geq 0$.

Ex. 34. Under the assumptions of the last exercise, $H_n(X, A) \simeq H_n(B)$.

Ex. 35. Show that $H_0(X, A) = 0$ if $A \neq \emptyset$.

Ex. 36. Show that $H_q(X, x_0) \simeq \begin{cases} \{0\} & \text{for } q = 0, \\ H_q(X) & \text{for } q \geq 1. \end{cases}$

Ex. 37. Use Mayer-Vietoris sequence to compute $H_*(\mathbb{S}^n)$ for all $n \geq 1$.

Ex. 38. Let X be S^1 along with any of its diameters. Compute its homology groups.

Ex. 39. Let X be the topologists' sine curve Exer. 31. The closed sine curve Y is the space X along with an arc c which intersect X only at the points $(0, -1)$ and $(1, \sin 1)$. Compute $H_*(Y)$ using a suitable Mayer-Vietoris sequence.

Ex. 40. Prove by two different methods that $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic iff $m = n$: i) their one point compactifications are not homeomorphic ii) the complement of a point in $\mathbb{R}^m$ and that its corresponding point in $\mathbb{R}^n$ are not homeomorphic.

Ex. 41. Assume that $X = U \cap V$, U and V open. Assume further that $U \cap V$ is nonempty and contractible. Compute the homology groups of X in terms of those of U and V.

Ex. 42. Assume that $X = A \cup B$, A and B closed and $A \cap B = \{x_0\}$. Assume further that x_0 has an open neighbourhood N in X such that $N \cap A$ and $N \cap B$ are contractible in such a way that during the contractions x_0 remain fixed. Compute the homology groups of X in terms of those of A and B.

Ex. 43. Compute $H_*(\mathbb{D}^n, \mathbb{S}^{n-1})$.

Ex. 44. Compute $H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$.

Ex. 45. Compute $H_*(\mathbb{S}^n)$ using the excision and homotopy theorems.

Ex. 46. If X is a Hausdorff space and $x \in X$, then $H_p(X, X \setminus \{x\})$ is called the p-th *local homology group* of X at x. If U is any neighbourhood of x_0 then $H_*(U, U \setminus \{x_0\}) \simeq H_*(X, X \setminus \{x_0\})$. (Thus $H_*(X, X \setminus \{x_0\})$ depends only on arbitrarily small neighbourhoods of x in X. Hence the epithet "local" is justified!) *Hint:* Excision theorem.

Ex. 47. Determine the local homology groups at various points of $\mathbb{D}^n$.

Ex. 48. Invariance of dimension. Let $x \in U \subset \mathbb{R}^m$ and $y \in V \subset \mathbb{R}^n$. Assume U and V are open in their respective spaces and that $(U, \{x\})$ is homeomorphic to $(V, \{y\})$. Then $m = n$. (This trivially follows from invariance of domain. You are required to use local homology groups, Exer. 44.)

Ex. 49. Invariance of the boundary. Let $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_1 \geq 0\}$. Let x, $y \in \mathbb{R}^n_+$ have neighbourhoods U and V such that $(U, x) \simeq (V, y)$. Then either both x and y lie on the boundary $\partial\mathbb{R}^n_+$ or both lie in the interior. *Hint:* If $x \in \partial\mathbb{R}^n_+$ then $(\mathbb{R}^n_+, \mathbb{R}^n_+ \setminus x)$ is contractible to (z, z) with z in the interior of $\mathbb{R}^n_+$. Hence $H(\mathbb{R}^n_+, \mathbb{R}^n_+ \setminus x) \simeq H(z, z) = 0$. If y lies in the interior then $H_n(\mathbb{R}^n_+, \mathbb{R}^n_+ \setminus y) \simeq H_n(\mathbb{R}^n, \mathbb{R}^n \setminus y)$ by excision.

Ex. 50. 101 Prove (once again!) that any homeomorphism of $\mathbb{D}^n$ onto itself maps $\mathbb{S}^{n-1}$ onto $\mathbb{S}^{n-1}$.

Ex. 51. Let $f\colon \mathbb{D}^n \to \mathbb{R}^n$ be continuous. Then there exists an $x \in \mathbb{D}^n$ such that $f(x) = 0$ or a $y \in \mathbb{S}^{n-1}$ such that $f(y) = \lambda y$ for some $\lambda \in \mathbb{R}$. *Hint:* Consider

$$g(x) := \begin{cases} 2(\|x\| - 1)x - (2 - 2\|x\|)f(x/\|x\|) & \text{for } \|x\| \geq 1/2, \\ -f(4\|x\|x) & \text{for } \|x\| \leq 1/2. \end{cases}$$

Then $g(z) = z$ for $z \in \mathbb{S}^{n-1}$ and g vanishes at some point in the interior of $\mathbb{D}^n$.

Ex. 52. Let $f\colon \mathbb{D}^n \to \mathbb{R}$ be continuous. Then $f(x) = x$ for some $x \in \mathbb{D}^n$ or $f(z) = \lambda z$ for some $z \in \mathbb{S}^{n-1}$ and $\lambda > 1$. *Hint:* Replace f by $f - Id$ in the last exercise.

Ex. 53. Let X be any space. Consider ΣX the suspension of X: it is the quotient of $X \times I$ induced by an equivalence relation whose equivalence classes are $X \times \{0\}$, $X \times \{0\}$ and the singletons (x, t) as $x \in X$ and $0 < t < 1$. Show that $\tilde{H}_{p+1}(\Sigma X) \simeq \tilde{H}_p(X)$.

Ex. 54. Show that $\Sigma S^n \simeq S^{n+1}$. Use this to compute $H_*(S^n)$.

Ex. 55. If $p \in S^n$ then $H_r(X \times S^n, X \times \{p\}) \simeq H_{r-n}(X)$. *Hint:* Express S^n as the union of two hemispheres and induct on n.

Ex. 56. If $q \in Y$ the homology exact sequence of $(X \times Y, X \times q)$ breaks up into short exact sequnces that split.

Ex. 57. Prove that $H_p(X \times S^n) \simeq H_{p-n}(X) \oplus H_p(X)$.

Ex. 58. Compute the homology of $S^m \times S^n$.

Ex. 59. Given $f\colon \mathbb{S}^n \to \mathbb{S}^n$ there is an associated map $\Sigma f\colon \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$ called the suspension of f. More specifically, let $\Sigma f(x,t) := \begin{cases} (x,t) & \text{if } x = 0 \\ (\|x\| \cdot f(x/\|x\|, t) & \text{if } x \neq 0. \end{cases}$ Show that the degree of Σf is that of f. *Hint:* Go through the computation of the degree of the map $(x_1, x_2, \ldots, x_{n+1}) \mapsto (-x_1, x_2, \ldots, x_{n+1})$.

Ex. 60. Show that given any integer n there is a map $f\colon \mathbb{S}^n \to \mathbb{S}^n$ with degree n.

Ex. 61. Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be of nonzero degree. Show that f is onto.

Ex. 62. Show that an even dimensional sphere $\mathbb{S}^n$ has no nowhere vanishing tangent vector field. *Hint:* Let u be a unit tangent vector field. Then $H(t,x) := \cos \pi t x + \sin \pi t u(x)$ is a homotopy between the identity and the antipodal map.

Ex. 63. Let f, $g\colon \mathbb{S}^n \to \mathbb{S}^n$ be such that $f(x) \neq g(x)$ for all $x \in \mathbb{S}^n$. Then f and $A \circ g$ are homotopic. (A is the antipode map.)

Ex. 64. If $f\colon \mathbb{S}^{2n} \to \mathbb{S}^{2n}$ is a map then there exists an x such that either $f(x) = x$ or $f(x) - x$. *Hint:* Use the last exercise. Or assuming the contrary observe that $\{x, f(x)\}$ spans a two dimensional vector space and hence we can find a nowhere vanishing tangent vector field. Apply Exer. 62.

Ex. 65. If n is even any map $\mathbb{P}^{2n} \to \mathbb{P}^{2n}$ has a fixed point.

Ex. 66. There is no continuous $f\colon \mathbb{S}^{2n} \to \mathbb{S}^{2n}$ such that x and $f(x)$ are orthogonal for all x. Use this to give another proof of Exer. 62.

Ex. 67. Let A be an $n \times n$ orthogonal matrix. Then A, as a linear map on $\mathbb{R}^n$ induces a map on $\mathbb{S}^{n-1}$. What is its degree?

Ex. 68. Let A be a nonsingular $n \times n$-matrix. Consider this as a linear map on $\mathbb{R}^n$. This extends to a continuous map f on $\mathbb{S}^n$. The degree of f is the sign of the determinant of A.

Ex. 69. Using the degree theory prove the fundamental theorem of algebra: *Every polynomial* $p(z) := z^n + c_1 z^{n-1} + \cdots + c_n$, $n > 0$ *has a zero. Hint:* If p has no zeros on $\mathbb{S}^1$, define $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ by $f(z) := \frac{p(z)}{|p(z)|}$. Then i) if p has no zeros in $|z| \leq 1$ then degree of f is zero and ii) if f has no zeros in $|z| \geq 1$ then degree of f is n.

Ex. 70. If a complex polynomial p has no zeros on $\mathbb{S}^1$ and has m zeros inside $\mathbb{S}^1$ (counted with multiplicity) then the map $f(z) := \frac{p(z)}{|p(z)|}$ has degree m.

Ex. 71. Let $\mathbb{D}$ be the closed unit ball in $\mathbb{R}^n$. Show that any homeomorphism of $\mathbb{D}$ onto itself carries $\mathbb{S}^{n-1}$ onto $\mathbb{S}^{n-1}$.

Ex. 72. 1 Show that $\mathbb{S}^n$ is not homeomorphic to any proper subset of itself.

Ex. 73. Prove that a continuous map $f\colon \mathbb{S}^n \to \mathbb{R}^n$ cannot be 1-1.

Ex. 74. Prove that the invariance of domain holds if the ambient space $\mathbb{S}^n$ is replaced by $\mathbb{R}^n$.

Ex. 75. Let A and B be homeomorphic subsets of $\mathbb{R}^n$. True or false? If A is closed then B is closed.

Ex. 76. Show that the invariance of domain is false if $\mathbb{S}^n$ is replaced by $\mathbb{D}^n$ for $n \geq 1$.

Ex. 77. Show that the invariance of domain holds for X as an ambient space then it holds for Y as an ambient space if Y is homeomorphic to X.

Ex. 78. Let $U \subset \mathbb{R}^n$ be open. Let $f\colon U \to \mathbb{R}^n$ be continuous and 1-1. Then f is a homeomorphism of U onto $f(U)$.

Ex. 79. Let $m < n$. Show that no subset of $\mathbb{S}^m$ can be homeomorphic to I^n.

Ex. 80. Let X be a topological space. Assume that X can be made into an n-dimensional manifold as well as an m-dimensional manifold. What can you say about m and n?

Ex. 81. Assume that X and Y are n-dimensional manifolds. Let $U \subset X$ and $V \subset Y$ be homeomorphic. If U is open in X then V is open in Y.

Ex. 82. Show that there is no continuous 1-1 map from an open set in $\mathbb{R}^n$ into $\mathbb{R}^m$ if $m < n$.

Ex. 83. Find an example of a map $f\colon \mathbb{R} \to \mathbb{R}^2$ which is 1-1 and continuous but for which the image of some open subset of $\mathbb{R}$ is not open in $f(\mathbb{R})$. *Hint:* Figure ∞!

Ex. 84. Show that $\mathbb{S}^n$ cannot be disconnected by removing a k-cell ($\simeq I^k$) for $0 \leq k \leq n$. *Hint:* This is an immediate corollary of a result needed for the Jordan-Brouwer separation theorem.

Ex. 85. Let $A \subset \mathbb{R}^n$ be homeomorphic to I^k ($0 \leq k \leq n$). Determine the homology groups of $\mathbb{R}^n \setminus A$.

Ex. 86. Let $n > 1$. Let A be a subset of $\mathbb{R}^n$ homeomorphic to $\mathbb{S}^{n-1}$. Then $\mathbb{R}^n \setminus A$ has precisely — components with A as a common boundary.

Ex. 87. Let A be a closed subset of $\mathbb{R}^n$ which is homeomorphic to $\mathbb{R}^{n-1}$. Show that $\mathbb{R}^n \setminus A$ has exactly — components.

Ex. 88. Let $A \subset \mathbb{R}^n$ be a closed subset homeomorphic to $\mathbb{R}^m$ ($m < n$). Compute $H_*(\mathbb{R}^n \setminus A)$.

Ex. 89. Let A and B be subsets of $\mathbb{S}^n$. Assume that A (resp. B) is homeomorphic to $\mathbb{S}^p$ (resp. $\mathbb{S}^q$), $0 < p \leq q \leq n$. Determine the homology groups of $\mathbb{S}^n \setminus (A \cup B)$ where i) $A \cap B = \emptyset$, ii) $|A \cap B| = 1$.