Existence of Continuous Logarithms

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

Proposition 1. *Let X be any topological space and let $F\colon X \times [0,1] \to \mathbb{C}^*$ be a continuous map. Assume that the restriction of F to $X \times \{0\}$ has a continuous logarithm, say, $\varphi\colon X \times \{0\} \to \mathbb{C}$ such that $\exp(\varphi(x,0)) = F(x,0)$. Then there exists a continuous logarithm ψ of F that extends φ.*

Proof. We first establish a local version of this result, that is, for each $x \in X$, there exists an set $U_x \ni x$ in X and a continuous map $\psi_x\colon U_x \times [0,1] \to \mathbb{C}$ with the properties:
(i) $\psi_x(x',0) = \varphi(x)$ for all $x \in U_x$,
(ii) $\exp(\psi_z(x',t)) = F(x',t)$ for $(x',t) \in U_x \times [0,1]$.

Given $(x_0,t) \in X \times [0,1]$, we let $V := \mathbb{C} \setminus (-\infty, 0)$. Then we know that there exists a continuous logarithm on V. By continuity of F, there exists an open set $U_t \ni x_0$ and $J_t \supseteq (t-\varepsilon_t, t+\varepsilon_t)$ such that $F(x,t) \in V$ for $(x,t) \in U_t \times J_t$. Now, $\{J_t : t \in [0,1]\}$ is an open cover of the compact set $[0,1]$. Let $0 = t_0 < t_1 \cdots < t_n = 1$ be points such that $\{J_k : 0 \le k \le n\}$ is a finite subcover of $[0,1]$. (Here, we have let J_k stand for J_{t_k}. Also, once a finite set of points are obtained by the compactness, we may assume if necessary by including, that 0 and 1 are among the finite set of points.) We now define $U := \cap_{k=0}^n U_k$, in an obvious notation. Clearly, $F(U \times [t_k, t_{k+1}]) \subset V$, $0 \le k \le n$. We now define the maps $\psi_k\colon U \times [t_k, t_{k+1}] \to \mathbb{C}$ inductively as follows.

Let $\log_0$ be the continuous logarithm on V. Then by composing the restriction of F to $U \times [0, t_1]$ with $\log_0$, we get a continuous logarithm of F, say, g_0: $g_0(x,t) := \log_0 \circ F(x,t)$. Since

$$\exp(g(x,0)) = F(x,0) = \exp(\varphi(x)),$$

we see that $\exp[g(x,0) - \varphi(x)] = 1$. Since we would like our logarithm of F to extend φ, we set

$$\psi_0(x,t) = g_0(x,t) - g(x,0) - \psi(x),$$

for $(x,t) \in U \times [0,t_1]$. It is then easily verified that $\exp(\psi_0) = F$ and $\psi(x,0) = \varphi(x)$ for $x \in U$.

It should be clear now how to define $\psi_i\colon U \times [t_1, t_2]$. Arguing in a way similar to 0-th level construction, we can find continuous functions $\psi_k\colon U \times [t_k, t_{k+1}] \to \mathbb{C}$ such that $\exp(\psi_k) = F$ and $\psi_k(x, t_k) = \psi_{k-1}(x, t_k)$. By gluing lemma, we get a continuous function $\psi\colon U \times [0,1]$ with the required properties.

Now we can complete the proof. We use an obvious notation. For any $x_1, x_2 \in X$, and for any $x \in U_1 \cap U_2$, we have

$$\exp(\psi_1(x,t)) = F(x,t) = \exp(\psi_2(x,t)),$$

so that we deduce that

$$\psi_1(x,t) - \psi_2(x,t) = 2\pi i n(t), \text{ for some } n(t) \in \mathbb{Z}.$$

The left side is a continuous function of t and hence $t \mapsto n(t)$ must also be continuous. Since $[0,1]$ is connected, we conclude that $t \mapsto n(t)$ must be a constant. Since $\psi_1(x,0) = \psi_2(x,0) = \varphi(x)$, we see that $n(0) = 0$. Hence $n(t) = 0$ for all $t \in [0,1]$.

What we have shown is that any pair of the functions ψ_x agree on their common domain. Note that the domains are open subsets of $X \times [0,1]$ and their union is $X \times [0,1]$. Hence, by gluing lemma, they yield a continuous function $\psi \colon X \times [0,1] \to \mathbb{C}$ with the required properties. □

Corollary 2. *Let $S \subset \mathbb{R}^n$ be any star-shaped subset. Then any continuous map $f \colon S \to \mathbb{C}^*$ has a continuous logarithm.*

Proof. Let $p \in S$ be a 'star'. Consider the map $F \colon S \times [0,1] \to \mathbb{C}^*$ defined by

$$F(x,t) := f(tx + (1-t)p) \text{ for } (x,t) \in S \times [0,1].$$

Since $F(x,0) = f(p) \neq 0$, if we take $\varphi(x) := \log_0(f(p))$, then φ is a continuous logarithm of $F(x,0)$. Hence by the last theorem, there exists a continuous logarithm, say, ψ of F. Clearly, $x \mapsto \psi(x,1)$ is a continuous logarithm of f. □

Corollary 3. *There exists no continuous retraction of the closed ball in $\mathbb{R}^2$ to its boundary, That is, there exists no continuous map $f \colon B[0,1] \to S^1$ with the property $f(z) = z$ for all $z \in S^1$.*

Proof. Suppose such a retraction f exists. Since $B[0,1]$ is convex, there exists a continuous logarithm, say, ψ of f. Restricting ψ to S^1 yields a continuous map with the property that $\exp(\psi(z)) = f(z) = z$. That is, the restriction of ψ to S^1 is a continuous argument on S^1. We know that this is impossible. This shows that there is no retraction of the ball to its boundary. □

From this corollary, one deduces Brouwer fixed point theorem for closed ball in the usual way.