A Continuous but Nowhere Differentiable Function

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The aim of this article is to construct (in a series of graded exercises) a Weierstrass type function on $\mathbb{R}$ which is continuous on $\mathbb{R}$ but not differentiable at any point.

1. Let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be differentiable at $x \in \mathbb{R}$. Let (s_n) and (t_n) be sequences such that $s_n \le x \le t_n$ with $\lim t_n - s_n = 0$, that is, $s_n \to x$ and $t_n \to x$. Show that the sequence $([\varphi(t_n) - \varphi(s_n)]/(t_n - s_n)$ is convergent.

   Let $s_n \le c \le t_n$ with $t_n - s_n \to 0$. Observe that for any $\alpha \in \mathbb{R}$, we have

   $$\frac{\varphi(t_n) - \varphi(s_n)}{t_n - s_n} - \alpha = \left(\frac{\varphi(t_n) - \varphi(c)}{t_n - c} - \alpha\right)\left(\frac{t_n - c}{t_n - s_n}\right) + \left(\frac{\varphi(a) - \varphi(s_n)}{c - s_n} - \alpha\right)\left(\frac{c - s_n}{t_n - s_n}\right).$$

   Now take $\alpha = f\varphi'(c)$, observe that $\left(\frac{t_n - c}{t_n - s_n}\right) \le 1$, $\left(\frac{c - s_n}{t_n - s_n}\right) \le 1$. Hence

   $$|\frac{\varphi(t_n) - \varphi(s_n)}{t_n - s_n} - \varphi'(c)| \le |\frac{\varphi(t_n) - \varphi(c)}{t_n - c} - \varphi'(c)| + |\frac{\varphi(a) - \varphi(s_n)}{c - s_n} - \varphi'(c)|.$$

   Now take limits to obtain the result.

   2nd Proof. Since f is differentiable at a, given $\varepsilon > 0$, there exists a $\delta > 0$ such that

   $$|f(x) - f(a) - f'(a)(x - a)| < \varepsilon \text{ for } 0 < |x - a| < \delta.$$

   Hence we have

   $$\begin{aligned} \varepsilon(x - a) \ &< f(x) - f(a) - f'(a)(x - a) \ < \varepsilon(a - x) \\ \varepsilon(a - y) &< f(y) - f(a) - f'(a)(y - a) \ < \varepsilon(y - a) \end{aligned}$$

   Subtracting the middle term of the second from that of the first the first, and using the inequalities in the correct way, we obtain

   $$-\varepsilon(x - y) < f(x) - f(y) - f'(a)(x - y) < \varepsilon(x - y).$$

   This yields the result.

   Third Proof. Recall that if f is differentiable at a, then there exists a function f_1 which is continuous at a with (i) $f_1(a) = f'(a)$ and (ii) $f(x) = f(a) + f_1(x)(x - a)$ for

$x \in J$. Apply (ii) with $x = x$ and $x = y$. Subtract the second from the first and after a simple algebraic manipulation we deduce for $x < a < y$,

$$f(x) - f(y) = f_1(x)(x - y)\,[f_1(x) - f_1(y)]\,(y - a)$$

It follows that for $x_n < a < y_n$ with $y_n - x_n \to 0$,

$$|\frac{f(x_n) - f(y_n)}{x_n - y_n} - f_1(x_n)| \leq |f_1(x_n) - f_1(y_n)||\frac{y_n - a}{y_n - x_n}| \leq |f_1(x_n) - f_1(y_n)|.$$

Letting $n \to \infty$, we obtain the required result.

2. A rational number of the form $m/2^n$ is called a *dyadic rational*. Show that the set D of dyadic rationals is dense in $\mathbb{R}$. Hence conclude that given $x \in \mathbb{R}$ and $n \in \mathbb{N}$, there exist dyadic rationals s_n, t_n such that $s_n = j/2^n$ and $t_n = (j+1)/2^n$ and $s_n \leq x < t_n$.

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := \min\{d(x, [x]), d(x, [x]+1)\}$, where $[x]$ stands for the greatest integer less than or equal to x. Draw the graph of f. Show that it is continuous on $\mathbb{R}$. (Observe that $f(x) = d(x, \mathbb{Z})$.) Note that $0 \leq f(x) \leq 1/2$ for all x.

4. Show that f is periodic of period 1, that is, $f(x) = f(x+1)$ for any $x \in \mathbb{R}$.

5. Consider the half intervals $[0, 1/2]$ and $[1/2, 1]$. Note that if s, t, say, $s < t$, lie in the same half interval, then the difference quotient $[f(t) - f(s)]/(t - s)$ is the slope of the chord joining $(s, f(s))$ and $(t, f(t))$ and it is either 1 or -1.

6. Let $f_0 = f$ and $f_n(x) := 2^{-n} f(2^n x)$ for $n \geq 1$. Draw graphs of f_0, f_1, f_2 on the 'same graph sheet'. Observe that $0 \leq f_n(x) \leq 2^{-(n+1)}$.

7. Let $g(x) := \sum_{k=0}^{\infty} f_k(x)$. Show that g is a continuous function on $\mathbb{R}$.

8. Let $s = m2^{-n}$ be a dyadic rational. Then for any $k \geq n$, we have $f_k(s) = 0$. Hence conclude that $g(s) = \sum_{k=0}^{n-1} f_k(s)$.

9. Let $x \in \mathbb{R}$ be fixed. Choose dyadic rationals s_n, t_n as in Item 2. Let $q_n := g(t_n) - g(s_n)/(t_n - s_n)$. In view of Item 3, to show that g is not differentiable at x, it is enough to show that the sequence $([g(t_n) - g(s_n)]/(t_n - s_n))$ does not converge.

   Hint: Write $s_n = j/2^n$ so that $t_n = (j+1)/2^n$. For $0 \leq k < n$, observe that $2^k(t_n - s_n) < 1/2$. Hence $2^k s_n$ and $2^k t_n$ lie in the same half intervals of $[0, 1]$. Use this to show that $q_n = \sum_{k=0}^{n-1} \pm 1$. Conclude that (q_n) cannot converge.

10. Conclude that g is continuous on $\mathbb{R}$ but not differentiable at any point of $\mathbb{R}$.