Curve Tracing
(Plotting the Graph of a Function)

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A more appropriate title would be "drawing graphs of functions". Whenever we see a function, we think of its graph, as this is more geometric. In elementary mathematics, we draw the graph of a function by plotting some points $(x, f(x))$ and joining them via straight lines and 'smooth' curves. This is a crude way of plotting curves. A much more refined method is to use (results of) differential calculus. In fact, calculus allows us to choose a good set of points $(x, f(x))$ and also tells us how to connect these points by appropriate smooth curves.

1 Preliminaries

In this section we shall collect some basic definitions and results which are needed for our purpose.

Let I or J stand for an interval in $\mathbb{R}$. We all know the significance of the sign of the derivative of $f: I \to \mathbb{R}$. If $f'(c) > 0$ for some $c \in I$, then f is increasing in some open interval (not necessarily I!) containing c. A point c in the open interval I is said to be a *critical point* of f if $f'(c) = 0$. We then know that such a point is a *relative* (or local) *maximum* (respectively *minimum*) if $f''(c) < 0$ (respectively if $f''(c) > 0$).

We wish to exploit the second derivative of f so that we get more information about the graph of a function.

Definition 1 Let $f: (a, b) \to \mathbb{R}$ be differentiable and $c \in (a, b)$. Let $\ell(x) := f'(c)(x - c) + f(c)$ be the tangent line to f at the point $(c, f(c))$.

Definition 2 The graph of f (or f, in short) is said to be *convex* (respectively *concave*) at $(c, f(c))$ (or at c) if there is an open interval J containing c such that for all $x \in J$, $x \neq c$,

$$f(x) > \ell(x) \qquad \text{(respectively } f(x) < \ell(x)).$$

See Figure 1. Convex is also known as "concave upward" and concave as "convex downward" in some old-fashioned books.

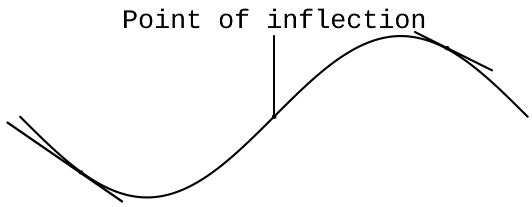

Figure 1: Convexity, Concavity and Points of Inflection

Geometrically this means that the points on the graph near to $(c, f(c))$ are above or below the tangent line at that point.

Definition 3 The point $(c, f(c))$ or c is said to be a *point of inflection* of f if there exists an open interval I containing c such that if $x \in I$, then either

$$f''(x) < 0 \quad \text{if } x < c \quad \text{and } f''(x) > 0 \quad \text{if } x > c, \tag{1}$$

or

$$f''(x) > 0 \quad \text{if } x < c \quad \text{and } f''(x) < 0 \quad \text{if } x > c. \tag{2}$$

See Figure 1.

Loosely speaking, a point of inflection is a point where f'' changes its sign. Geometrically this means that the graph changes from convexity to concavity or vice versa at that point. There is also a physical interpretation: If we consider $x := f(t)$ as the position of a moving particle, then $x'(t) = f'(t)$ is the velocity and $x''(t) = f''(t)$ is the acceleration of the particle. We say the particle is *accelerating* (respectively *decelerating*) at time t is $f''(t) > 0$ (respectively $f''(t) < 0$). Thus a point of inflection is the time at which the particle switches from acceleration to deceleration or vice versa.

There is another definition which is also geometric though not equivalent to the one given above: c is a point of inflection if the tangent line at c crosses the graph of the function.

We would like to caution the reader that there is no uniform agreement on the correct definition of an inflection point. I have chosen this, as this is very geometric and least technical and can be taught at the first year B.Sc class.

One then has the following theorem.

Theorem 1 *Let $f\colon (a,b) \to \mathbb{R}$ be differentiable. Assume that $f''(c)$ exists at $c \in (a,b)$. Then*

(1) *If $f''(c) > 0$, then f is convex at c.*

(2) *If $f''(c) < 0$, then f is concave at c.*

(3) *If c is point of inflection, then $f''(c) = 0$.*

$\square$

Proof The proof uses mean value theorem.

i) To show that the graph of f is convex at c, we need to show that $f(x) - \ell(x) > 0$ for all $x \neq c$ in an open interval containing c. We have

$$f(x) - \ell(x) = f(x) - f(c) - f'(c)(x-c). \tag{3}$$

See Figure 2.

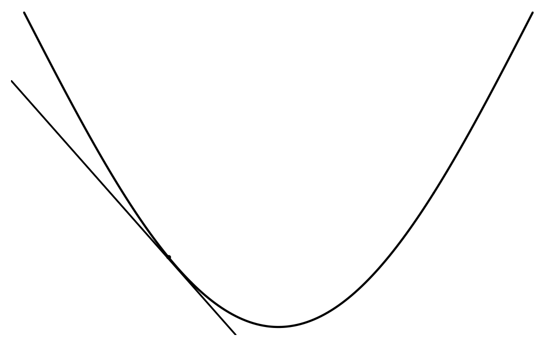

Figure 2:

We first apply the mean value theorem to f either on $[c,x]$ if $x > c$ or on $[x,c]$ if $x < c$. In any case, there exists a point x_1 between x and c such that

$$f(x) - f(c) = f'(x_1)(x-c).$$

Substituting Equation () in Equation (3) gives

$$\begin{aligned} f(x) - \ell(x) &= [f'(x_1) - f'(c)]\,(x-c) \\ &= \frac{f'(x_1) - f'(c)}{x_1 - c}(x_1 - c)(x-c). \end{aligned}$$

Now if $f''(c) > 0$, the difference quotient $\frac{f'(x_1)-f'(c)}{x_1-c}(x_1-c)$ must be positive for all $x_1 \neq c$ in some open interval I containing c. (Why?) But if x_1 is between x and c $(x_1-c)(x-c) > 0$. Hence for any $x \in I$, $x \neq c$, we have $f(x) - \ell(x) > 0$. Thus (1) is proved.

(2) is proved similarly. (3) follows from (1) and (2).

$\square$

Remarks 1

1. The vanishing of $f''(c)$ is not sufficient for c to be a point of inflection. For instance, consider $f(x) := x^4$. f is convex at 0.

2. f may have a point of inflection according to our definition but the second derivative may fail to exist. Consider $f(x) := x^{\frac{1}{3}}$. Or, consider

$$g(x) := \begin{cases} x^2 & \text{for } x < 0, \\ x^{1/2} & \text{for } x \geq 0. \end{cases}$$

Example 1 Let $f(x) := x^3 - 6x^2 + 9x$. Then $f'(x) = 3(x-1)(x-3)$ and $f''(x) = 6x - 12$. $f''(x) > 0$ if $x > 2$ and it is less than $x < 2$. Hence f is convex for $x > 2$ and concave for $x < 2$. $x = 2$ is a point of inflection.

To sketch the graph of f, we determine the intervals on which f is monotone. Since $f'(x) = 3(x-1)(x-3)$, $f'(x) > 0$ is $x < 1$ or if $x > 3$. Hence f is increasing on $(-\infty, 1)$ and $(3, \infty)$. If $x \in (1, 3)$, then f is decreasing on $(1, 3)$. Also, $f'(1) = 0 = f'(3)$ and $f''(1) < 0$ and $f''(3) > 0$ so that $f(1) = 4$ is a relative maximum and $x = 3$ is a relative minimum with the minimum $0 = f(3)$. See Figure 3.

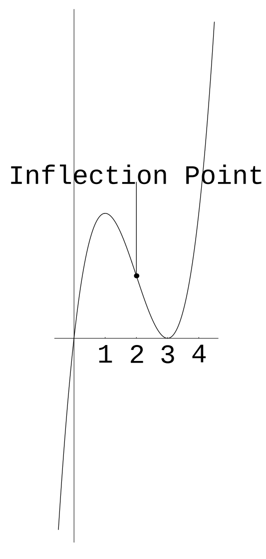

Figure 3: Graph of $x^3 - 6x^2 + 9x$

Exercise 1 Determine the intervals of convexity, concavity and the points of inflection for the following functions:

(i) $f(x) = x^3 + 9x$.

(ii) $f(x) = x/(x^2 - 1)$.

(iii) $f(x) = e^{-x^2}$.

(iv) $f(x) = \sqrt{1 + x^2}$.

(v) $f(x) = 2x^3 - 3x^2 + 3x + 1$.

Exercise 2 Show that a polynomial of degree n can have at most $n-2$ points of inflection.

Exercise 3 If $f(x) = ax^3 + bx^2$, determine a and b so that the graph of f will have a point of inflection at $(1, 2)$.

One more definition is needed if we want to discuss the behaviour of the graph as $x \to \pm\infty$.

Before we define the next concept, we invite the reader to draw the graphs of the following functions: $f(x) = 1/(x-a)$ and $g(x) = 1/(x-a)^2$ for $x \neq a$. Check your pictures with Figure 4 and Figure 5.

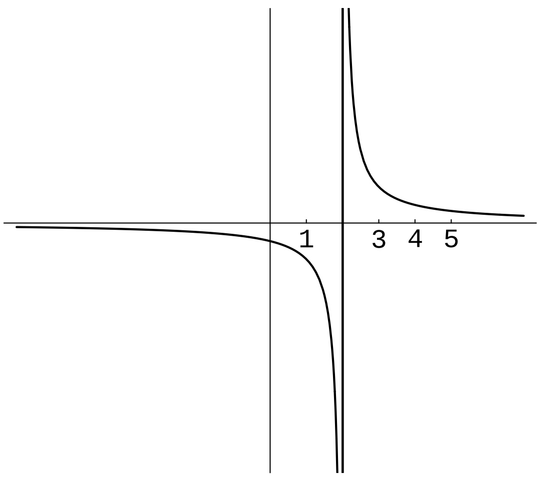

Figure 4: Graph of $1/(x - 2)$

Definition 4 An *affine* (also called "linear" in analysis) function $\ell(x) := ax + b$ is called an *asymptote* for f as $x \to \infty$ if

$$\lim_{x \to \infty} (f(x) - \ell(x)) = 0.$$

A vertical line $x = a$ is an aymptote for f if $\lim_{x \to a+} f(x)$ is infinite or $\lim_{x \to a-} f(x)$ is infinite.

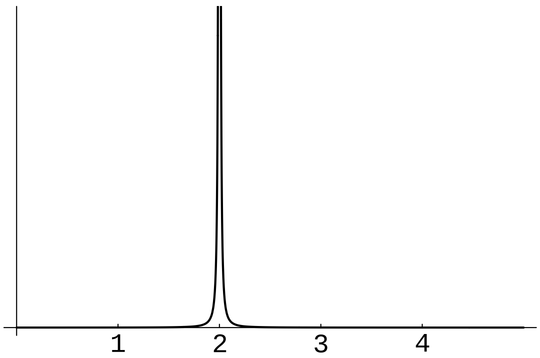

Figure 5: Graph of $1/(x-2)^2$

2 Examples *cum* Exercises

Let $f(x) := p(x)/q(x) = (a_n x^n + \cdots + a_0)/(b_m x^m + \cdots + b_0)$ be a rational function.

- If degree of $p(x) = n < m =$ degree of $q(x)$, then f has the x-axis as an asymptote.
- If $n = m$, then f has a horizontal asymptote $\ell(x) = a_n/b_m$.
- If $n = m+1$, we then divide p by q and write $f(x) = ax + b + c(x)$ where $\lim_{x\to+\infty} c(x) = 0 = \lim_{x\to-\infty} c(x)$. The line $\ell(x) := ax + b$ is a linear asymptote for f.
- If $n > m+1$, then f has no non-vertical linear asymptote.
- The vertical asymptote for a rational function corresponds to the zeroes of the denominator.

The proof of these claims is an easy exercise in the theory of limits.

3 Some Tips on Curve Tracing

We list below some of the points we must bear in mind while attempting to draw the graph of a function.

1. Look for symmetries of the given function f: Is it odd or even? (i.e. $f(-x) = -f(x)$ or $f(-x) = f(x)$). Is it periodic, i.e., is $f(x+a) = f(x)$ for all x and a fixed a?
2. Look for the points $(0, f(0))$, $(x, f(x) = 0)$, i.e., the point one of whose coordinates lies on the coordinate axes.

3. Look for points (if any) at which f is not defined. In case such a point exists, study the behaviour of f near the point.

4. Determine the behaviour of f as x becomes large through positive values (respectively negative values). Find the asymptotes, if any.

5. Locate critical points, local maxima, local minima of f.

6. Find the intervals on which f is increasing or decreasing.

7. Determine the intervals on which f is concave or convex.

8. Locate the points of inflection of f.

We illustrate these points by means of examples.

Example 2 Let $f(x) = x^4 - x^2$.

1. $f(-x) = f(x)$ and hence f is even.

2. At $x = 0, f(x) = 0$. Also $f(x) = 0$ iff $x^2(x^2 - 1) = 0$ iff $x = 0$ or $x \pm 1$. Thus, at $x = 0, \pm 1$, the graph intersects the y-axis at $y = 0$.

3. f is defined for all values of x.

4. As $x \to +\infty$, $f(x) = x^2(x^2 - 1) \to \infty$. As $x \to -\infty$, $f(x) = (-x)^2((-x)^2 - 1) \to \infty$. To see this, we need only observe that if $|x| > 1$, $x^2 - 1 > 0$.

5. $f'(x) = 4x^3 - 2x = 2x(2x^2 - 1)$. The critical points of f are those x for which $f'(x) = 0$, i.e., $2x(2x^2 - 1) = 0$ or $x = 0$ or $x = \pm\frac{1}{\sqrt{2}}$. To understand the nature of critical points, we find: $f''(x) = 12x^2 - 2$ so that $f''(0) = -2 < 0$ and $x = 0$ is a local maximum. $f''(\pm\frac{1}{\sqrt{2}}) = 12.\frac{1}{2} - 2 = 4 > 0$ and $x = \pm\frac{1}{\sqrt{2}}$ is a local minimum.

6. We can therefore write

$$\mathbb{R} = (-\infty, -\frac{1}{\sqrt{2}}) \cup (\frac{-1}{\sqrt{2}}, 0) \cup (0, \frac{1}{\sqrt{2}}) \cup (\frac{1}{\sqrt{2}}, \infty) \cup \{\text{critical points}\}.$$

   $f''(x) = 12x^2 - 2 = 2(6x^2 - 1)$.
   $f''(x) > 0$ if $6x^2 - 1 > 0$ or $x^2 > \frac{1}{6}$.
   $f''(x) < 0$ if $6x^2 - 1 < 0$ or $x^2 < \frac{1}{6}$.
   That is, according as, $|x| > 1/\sqrt{6}$ or $|x| < 1/\sqrt{6}$. Hence f is convex on $(-\infty, -1/\sqrt{6})$ and $(1/\sqrt{6}, \infty)$ and is concave on $(-1/\sqrt{6}, 1/\sqrt{6})$.

7. Hence $x = \pm 1/\sqrt{6}$ are points of inflection. Note also that $f''(x) = 12x^2 - 2$. $f''(x) = 0 \iff 6x^2 = 1 \iff x = \pm 1/\sqrt{6}$. $f'''(x) = 24x$ so that $f'''(\pm 1/\sqrt{6}) \neq 0$.

With this information, we can plot the curve Figure 6.

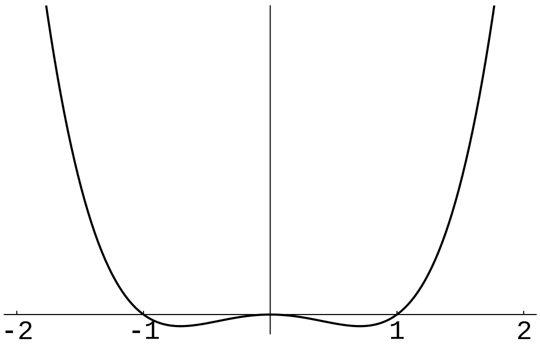

Figure 6: Graph of $x^4 - x^2$

Example 3 Let $f(x) = x + \frac{1}{x} = \frac{x^2+1}{x}$. Then

1. $f(-x) = \frac{(-x)^2+1}{-x} = -\frac{x^2+1}{x} = -f(x)$. Hence f is odd.

2. f is not defined at $x = 0$.

3. For x near 0 and greater than 0, $\frac{1}{x}$ is positive and large. Similarly for $x < 0$ and near 0, $\frac{1}{x}$ is negative and large. Hence $f(x) = x + \frac{1}{x}$ is large and positive for $x > 0$, x near 0 and large and negative for $x < 0$, x near 0.

4. $f'(x) = 1 - \frac{1}{x^2}$. $f'(x) > 0$ if $1 - \frac{1}{x^2} > 0$, or $1 > \frac{1}{x^2}$ or $x^2 > 1$ or $|x| > 1$. Thus f is increasing in $(-\infty, -1)$ and $(1, \infty)$. f is decreasing in $(-1, 0)$ and $(0, 1)$.

5. $f'(x) = 0$ iff $1 = \frac{1}{x^2}$ or $x^2 = 1$ or $x = \pm 1$. $f''(x) = 0 + 2\frac{1}{x^3}$. $f''(-1) < 0$ and $f''(1) > 0$. Thus $x = -1$ is a local maximum and $x = 1$ is a local minimum.

6. $f(x) = 0 \iff x + \frac{1}{x} = 0 \iff x = -\frac{1}{x} \iff x^2 = -1$. Therefore the graph of f does not intersect the y-axis.

7. $f''(x) = 2\frac{1}{x^3}$ and is less than 0 if $x < 0$ and greater than 0 if $x > 0$. Thus f is concave on $(-\infty, 0)$ and convex on $(0, \infty)$.

8. The line $y = x$ is a linear asymptote. The line $x = 0$ is a vertical asymptote.

With these facts we plot the curve. See Figure 7.

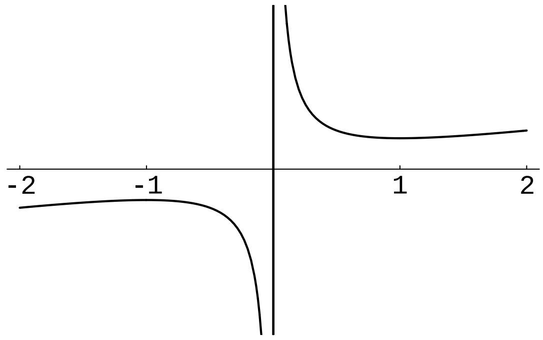

Figure 7: Graph of $x + 1/x$

Example 4 Let $f(x) = x^2/(x^2 - 1)$.

1. $f(x) = f(-x)$. Thus f is symmetric.
2. We have

$$\begin{aligned} f'(x) &= \frac{-2x}{(x^2-1)^2} \\ f''(x) &= \frac{2(3x^2+1)}{(x^2-1)^3} \end{aligned}$$

   Thus $f'(x) = 0$ iff $x = 0$ and $f''(0) < 0$. Hence $f(0) = 0$ is a relative maximum.

3. Assume $x > 0$. (Enough to consider this case thanks to symmetry!) Then

$$\begin{aligned} f''(x) &> 0 \text{ if } x > 1 \quad \text{ and} \\ f''(x) &< 0 \text{ if } 0 < x < 1. \end{aligned}$$

4. $\lim_{x\to\infty} f(x) = 1$.
5. Let us find the behaviour of f near $x = 1$. We note that $\frac{x^2}{x^2-1} = \frac{x^2}{x+1}\ \frac{1}{x-1}$ so that $\lim x \to 1_+ \frac{x^2}{x+1}\ \frac{1}{x-1} = \infty$ whereas $\lim x \to 1_- \frac{x^2}{x+1}\ \frac{1}{x-1} = -\infty$.

See Figure 8.

Example 5 Consider $f(x) = (x^2 - 1)/(x + 2)$.

1. $x = -2$ is a vertical asymptote.

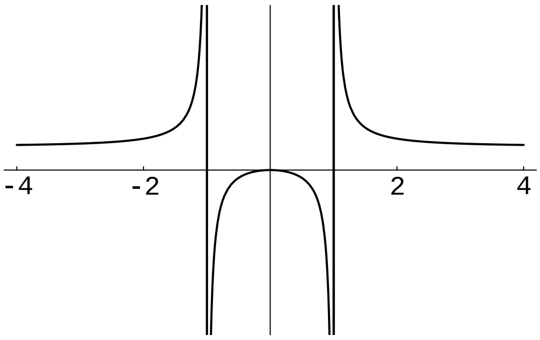

Figure 8: Graph of $f(x) = x^2/(x^2-1)$

2. $f(x) = x - 2 + (3/(x+2))$. Thus the line $\ell(x) = x - 2$ is a linear asymptote.

3. $f'(x) = \dfrac{x^2+4x+1}{(x+2)^2}$ so that $f'(x) = 0$ iff $x = -2 \pm \sqrt{3}$. $f''(x) = 6/(x+2)^3$ which is > 0 at $x = -2-\sqrt{3}$ and > 0 at $x = -2+\sqrt{3}$. Hence f has a relative maximum at $x = -2-\sqrt{3}$ and a minimum at $x = -2+\sqrt{3}$.

See Figure 9.

Example 6 Consider $f(x) = \sin^2 x$.

1. It is periodic on $[-\pi, \pi]$.

2. It is symmetric in x.

3. $f'(x) = 2\sin x \cos x = \sin 2x$ and $f''(x) = 2\cos 2x$. Hence $f(x) = 0$ iff $x = 0, \pm n\pi$, $f'(x) = 0$ iff $x = 0, \pm n\frac{\pi}{2}$ and $f''(x) = 0$ iff $x = \pm\frac{\pi}{4}, \frac{3\pi}{4}, \ldots, \pm\frac{n\pi}{4}, \ldots$. Further, $f''(0) = 2\cos 0 > 0$ implies that f attains a minimum at $x = 0$ and $f''(\pm\frac{\pi}{2}) = 2\cos(\pm\pi) < 0$ implies that f attains a maximum at $x = \pm\frac{\pi}{2}$. See Figure 10.

4. $f'''(x) = -4\sin 2x$. $f''(\pm\frac{\pi}{4}) = 0$ and $f'''(\pm\frac{\pi}{4}) = -(\pm\sin\frac{\pi}{2}) \neq 0$ implies $x = \pm\frac{\pi}{4}$ is an inflection point.

5. $f(-x) = f(x)$ implies f is even. $f'(x) = \sin 2x \geq 0$ iff $0 \leq 2x \leq \pi$ iff $0 \leq x \leq \frac{\pi}{2} \leq 0$ iff $\frac{\pi}{2} \leq x \leq \pi$. Hence f is increasing in $(0, \frac{\pi}{2})$ and decreasing in $(\frac{\pi}{2}, \pi)$.

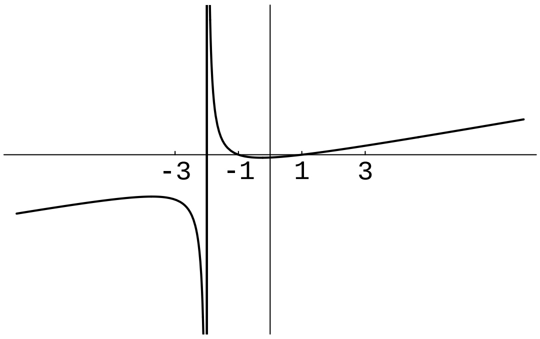

Figure 9: Graph of $f(x) = \frac{x^2-1}{x+2}$

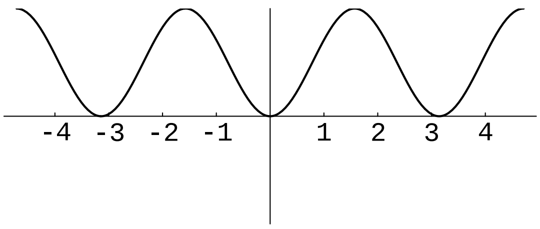

Figure 10: Graph of $\sin^2 x$