Decimal, Binary and Ternary Expansions

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1 Introduction

The geometric ideas behind the decimal, binary and ternary expansions of any real number depend on the nested interval theorem. We recall it in a form useful to us.

Theorem 1 (Nested Interval Theorem)**.** *Let $I_k \subset \mathbb{R}$ be finite intervals with endpoints a_k and b_k such that (i) $I_{k+1} \subset I_k$ for each $k \in \mathbb{N}$ and (ii) $\lim_{k\to\infty} \ell(I_k) = \lim_{k\to\infty} b_k - a_k \to 0$. Then there exists a unique $c \in \mathbb{R}$ such that $a_k \leq c \leq b_k$ for all k.*

Proof. Consider the set $A := \{a_k : k \in \mathbb{N}\}$. This set is nonempty, bounded above by each of b_n. Hence by the least upper bound property of $\mathbb{R}$, there exists $c \in \mathbb{R}$ such that $c = \sup A$. Then $c \leq b_n$, since c is the l.u.b. of A and each b_n is an upper bound for A. Also, since c is an upper bound for A, $a_n \leq c$ for all n. Thus, $a_n \leq c \leq b_n$.

If d is also such that $a_n \leq d \leq b_n$ for each n, then, $c, d \in [a_n, b_n]$ for each n. From this we conclude that $|c - d| \leq b_n - a_n$ for all n. As $b_n - a_n \to 0$, it follows that $|c - d| = 0$. Hence $c = d$. ☐

Remark 2. Let us reiterate that we did not assume that the intervals I_k are closed. However, our conclusion was that $c \in [a_n, b_n]$ for all n. We do **not** claim that $c \in I_n$. An easy example is $I_k = (0, 1/k)$.

How do we plan to use this? Let $p \in \mathbb{N}$ be greater than or equal to 2. Then we want to show that there exists p-expansion for any real number x in the following sense: There exists an integer x_0 and numbers a_k lying in $\{0, 1, \dots, p-1\}$ such that

$$x = x_0 + \sum_k \frac{a_k}{p^k}.$$

We may assume without loss of generality that $x \geq 0$. For, $x < 0$ and if $y = -x = y_0 + \sum_k \frac{a_k}{p^k}$ is the p-expansion of y, then $x = -(y_0 + 1) + \sum_{k=1}^{\infty} \frac{a_k}{p^k}$ is the p-expansion for x. Any real number x can be written in the form $x = x_0 + a$, where $x_0 \in \mathbb{Z}$ and $a \in [0, 1)$. This suggests that it suffices to consider only $x \in [0, 1)$.

Let $x \in [0, 1)$ be given. The key geometric idea is to subdivide $[0, 1)$ into p-equal parts, choose the one which contains our element x, then subdivide this subinterval into p-equal parts choose the one in which x lies and so on. We are mostly interested when $p = 2, 3, 10$. When $p = 2, 3, 10$, the expansions are respectively called *binary*, *ternary* and *decimal.*

2 Decimal Expansions

Definition 3. A *positive decimal form* is a series of the form

$$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} + \cdots$$

denoted by $a_0.a_1a_2\cdots a_n\cdots$, where $a_0 \in \mathbb{Z}^+$ and $a_n \in \{0, 1, \ldots, 9\}$ for each $n \in \mathbb{N}$.

Theorem 4. *Every positive decimal form converges to a positive real number. (We then say the decimal form represents the real number).*

Proof. Consider the positive decimal form $a_0.a_1a_2\cdots a_n\cdots$. Now

$$\begin{aligned} s_1 &= a_0 + \frac{a_1}{10} \leq a_0 + \frac{9}{10} \\ s_2 &= a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} \\ &\leq a_0 + \frac{9}{10} + \frac{9}{10^2} \\ &\vdots \qquad\qquad \vdots \end{aligned}$$

By induction we have

$$\begin{aligned} s_n &\leq a_0 + \frac{9}{10}\left(1 + \frac{1}{10} + \cdots + \frac{1}{10^{n-1}}\right) \\ &\leq a_0 + \frac{9}{10}\left(\frac{1}{1 - \frac{1}{10}}\right) = a_0 + 1. \end{aligned}$$

Then $\{s_n\}$ is an increasing sequence of positive reals and is bounded above by $a_0 + 1$. Hence $\{s_n\}$ is convergent to a real number $a \in \mathbb{R}$. (We then say $a_0.a_1a_2\ldots a_n\ldots$ represents a.) □

Theorem 5. *Every positive real number has a positive decimal representation.*

Proof. Given a real number $a \geq 0$, we construct a series as follows: Let a_0 be the greatest integer less than or equal to a. Let

$$\begin{aligned} &a_1 \quad \text{be the greatest integer such that} \quad a_0 + \frac{a_1}{10} \leq a \\ &\qquad\qquad \vdots \\ &a_n \quad \text{be the greatest integer such that} \quad a_0 + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n} \leq a < a_0 + \frac{a_1}{10} + \cdots + \frac{a_n + 1}{10^n} \end{aligned}$$

We claim that $a_0.a_1a_2\ldots a_n\ldots$ represents a. For this we need to show that

1. $a_0.a_1a_2\dots a_n\dots$ is a decimal form, i.e., $a_n \in \{0,1,\dots,9\}$ for all $n \in \mathbb{N}$.

2. $a = a_0 + \sum_{n=1}^{\infty} \frac{a_n}{10^n}$.

1. Now $a_0 + \frac{a_1}{10} \le a$. So $a_1 \ge 0$. Also, $\frac{a_1}{10} \le a - a_0 < 1$, so $a_1 < 10$ or $0 \le a_1 \le 9$. By induction, we find that $0 \le a_n \le 9$.

We prove by induction that we can find integers a_n, $0 \le a_n \le 9$ such that

$$\sum_{i=0}^{n-1} \frac{a_i}{10^i} + \frac{a_n}{10^n} \le a < \sum_{i=0}^{n-1} \frac{a_i}{10^i} + \frac{a_n+1}{10^n}. \tag{1}$$

This integer is denoted by a_n.

Assume $a_1,\dots,a_{n-1}$ are chosen for $n \ge 1$. Let a_n be the greatest integer k such that

$$\sum_{i=0}^{n-1} \frac{a_i}{10^i} + \frac{k}{10^n} \le a < \sum_{i=0}^{n-1} \frac{a_i}{10^i} + \frac{k+1}{10^n}.$$

holds. Then $a_n \ge 0$. Also $a_n < 10$. For, otherwise. $a_n \ge 10$, so that we can write $a_n = 10+a_n'$, with $a_n' \ge 0$. But then

$$\sum_{i=0}^{n-1} \frac{a_i}{10^i} = a_0 + \frac{a_1}{10} + \cdots + \frac{a_{n-1}}{10^{n-1}} + \frac{1}{10^{n-1}} + \frac{a_n'}{10^n} \le a.$$

In particular,

$$a_0 + \frac{a_1}{10} + \cdots + \frac{a_{n-1}+1}{10^{n-1}} \le a,$$

which contradicts our induction hypothesis on a_{n-1}.

2. $|a - (a_0 + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n})| < \frac{1}{10^n}$ and hence **2.** follows. □

The geometric meaning behind this construction of the decimal representation is as follows:

Given $a \ge 0$, let a_0 be as above. Then $x := a - a_0 \in [0,1)$. We subdivide $[0,1) = I$ into ten intervals I_k for $0 \le k \le 9$:

$$I_k := [\frac{k}{10}, \frac{k+1}{10}), \quad k = 0,1,\dots,9.$$

Then I is the disjoint union of the I_k's so that x belongs to precisely one of the I_k's. This k is called a_1. Thus $a_1 = \max\left\{k \in \mathbb{N} \;\middle|\; \frac{k}{10} \le x\right\}$.

Next we subdivide I_{a_1} into ten intervals each of length $\frac{1}{10^2}$. The intervals are $[\frac{a_1}{10} + \frac{k}{10^2}, \frac{a_1}{10} + \frac{k+1}{10^2})$, $k = 0,\dots 9$. a_2 is that integer k such that x lies in the interval determined by a_2. Thus

$$a_2 = \max\left\{k \in \mathbb{N} \;\middle|\; \frac{a_1}{10} + \frac{k}{10^2} \le x\right\}.$$

Recursively, we proceed to find a_k's:

1. a_1 is the largest integer k such that $\frac{k}{10} \leq x$.

2. After $a_1, \dots, a_n$ are defined, a_{n+1} is the largest integer k such that

$$\frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_n}{10^n} + \frac{k}{10^{n+1}} \leq x.$$

What we have done is to locate x successively in the semi-closed intervals J_n where

$$J_n = [\frac{a_1}{10} + \cdots + \frac{a_n}{10^n}, \frac{a_1}{10} + \cdots + \frac{a_n + 1}{10^n}).$$

Note that $\{J_n\}$ is a nested sequence of intervals and that $\cap J_n = \{x\}$. For, if $x, y \in \cap J_n$, then $|x - y| < 10^{-n}$ for all n.

The map from the decimal forms to the real numbers is not one-to-one. Some real numbers have two decimal representations: For example, both $0.500\dots$ and $0.499\dots$ represent $\frac{1}{2}$. First let us see how they arise. If we decide to partition the interval $(0, 1]$ (in stead of $[0, 10)$ using the subintervals $(i/10, (i + 1)/10]$ and choose a_1 to be largest integer i such that $x \in (i/10, (i + 1)/10]$ and proceed as earlier we shall get the expansions such as $0.4\overline{9}$ for $1/2$ in stead of 0.5. In analytical terms, we are looking for a_n's defined as

$$a_n = \text{ the greatest integer such that } a_0 + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n} < x \leq a_0 + \frac{a_1}{10} + \cdots + \frac{a_n + 1}{10^n}.$$

Theorem 6. *Two distinct positive decimal forms* $a_0 + \sum_{k=1}^{\infty} \frac{a_k}{10^k}$ *and* $\alpha_0 + \sum_{k=1}^{\infty} \frac{\alpha_k}{10^k}$ *represent the same positive real number iff there exists* $N \in \mathbb{N}$ *such that*

1. $a_k = \alpha_k$ *for* $0 \leq k \leq N$.

2. $a_{N+1} = \alpha_{N+1} - 1$ *(or* $\alpha_{N+1} = a_{N+1} - 1$*).*

3. $a_k = 9$ *and* $\alpha_k = 0$ *for* $k > N + 1$ *(*$\alpha_k = 9$ *and* $a_k = 0$ *for* $k > N + 1$ *respectively).*

Proof. Suppose that the two decimal forms have the specified properties. Then if

$$s_n := a_0 + \sum_{k=1}^{n} \frac{a_k}{10^k} \text{ and } \sigma_n := \alpha_0 + \sum_{k=1}^{n} \frac{\alpha_k}{10^k},$$

we have $\sigma_n - s_n \leq \frac{1}{10^n}$ for all n. Let $\sigma_n \to \alpha$. Then $\alpha - \sigma_n \leq \frac{1}{10^n}$ for all n and hence

$$\alpha_n - s_n = \alpha_n - \sigma_n + \sigma_n - s_n \leq \frac{2}{10^n}, \qquad \text{for all } n.$$

Hence $s_n \to \alpha$, and so the two decimal forms represent the same real number.

Conversely, assume that the two decimal forms represent the same real number a. We have $a - s_n \leq \frac{1}{10^n}$ and $a - \sigma_n \leq \frac{1}{10^n}$ for all n. This implies $|\sigma_n - s_n| \leq \frac{1}{10^n}$ for all n. Draw pictures.

Let $a_k = \alpha_k$ for $0 \leq k \leq n$ for some $n \in \mathbb{Z}^+$. Since $|\sigma_{n+1} - s_{n+1}| \leq \frac{1}{10^{n+1}}$, we have $|\alpha_{n+1} - a_{n+1}| = 0$ or 1. Suppose $a_{n+1} = \alpha_{n+1} - 1$. Then $\sigma_{n+1} = s_{n+1} + \frac{1}{10^{n+1}}$. But

$a \leq s_{n+1} + \frac{1}{10^{n+1}} = \sigma_{n+1}$. Therefore $a = \sigma_{n+1}$ and hence $\alpha_k = 0$ for $k > n+1$. Thus $a = s_{n+1} + \frac{1}{10^{n+1}} = \sigma_{n+1}$. To have $a - s_{n+2} \leq \frac{1}{10^{n+2}}$, we require $a_{n+2} = 9$. For,

$$\begin{aligned} \frac{1}{10^{n+2}} &\geq a - s_{n+2} \\ &= s_{n+1} + \frac{1}{10^{n+1}} - \left(s_{n+1} + \frac{a_{n+2}}{10^{n+2}}\right) \\ &= \frac{10}{10^{n+2}} - \frac{a_{n+2}}{10^{n+2}} = \frac{10 - a_{n+2}}{10^{n+2}}. \end{aligned}$$

Thus, we must have $10 - a_{n+2} \leq 1$ or $9 \geq a_{n+2} \geq 10 - 1 = 9$. Hence $a_{n+2} = 9$.

Also, $a = s_{n+1} + \frac{1}{10^{n+1}} = s_{n+2} + \frac{a_{n+2}}{10^{n+2}} + \varepsilon = s_{n+1} + \frac{9}{10^{n+2}} + \varepsilon$. So, $\varepsilon = \frac{1}{10^{n+2}}$. We thus find $a = s_{n+2} + \frac{1}{10^{n+2}}$. By induction, we have $a_k = 9$ for $k > n+1$. □

Thus we have shown that a real number α admits a dual decimal representation iff α is of the form $\frac{k}{10^n}$, for some $k \in \mathbb{Z} \setminus \{0\}$. In other words, we have proved:

The set of reals with dual decimal representation is the set of rational numbers $\frac{p}{q}$ (where p, q are relatively prime) where $q = 2^m 5^n$ for $m, n \in \mathbb{Z}^+$.

A *repeating* decimal expansion is one of the form

$$a_0.a_1a_2 \ldots a_n b_1 b_2 \ldots b_m b_1 b_2 \ldots b_m b_1 b_2 \ldots b_m \ldots$$

Theorem 7. *A real number is rational iff it has a (terminating) repeating decimal representation.*

Proof. It is sufficient to consider $x > 0$. Suppose $x \in \mathbb{Q}$, $x = \frac{p}{q}$, where $p, q \in \mathbb{N}$ are relatively prime. Now

$$\begin{aligned} p &= a_0 q + r_0 \\ 10 r_0 &= a_1 q + r_1 \\ &\vdots \\ 10 r_n &= a_{n+1} q + r_{n+1} \end{aligned}$$

where $r_i \in \{0, \ldots, q-1\}$.

We thus get a decimal representation for $x = \frac{p}{q}$ as $x = a_0.a_1a_2 \ldots a_n \ldots$. By pigeon -hole principle, at least two of the elements $r_0, \ldots, r_q$ are same as $r_n \in \{0, \ldots, q-1\}$. Suppose $r_s = r_t$ for some $s \in \{1, \ldots, q-1\}$ and $t \in \{0, \ldots, s-1\}$. We then have a repetition in the decimal representation.

Conversely, suppose x has a repeating decimal representation

$$x = a_0.a_1 \ldots a_n b_1 \ldots b_m b_1 \ldots b_m \ldots.$$

Then

$$\begin{aligned} x &= a_0 + \sum_{k=1}^{n} \frac{a_k}{10^n} + \left(\frac{b_1}{10^{n+1}} + \cdots + \frac{b_m}{10^{n+m}} \right) + \left(\frac{b_1}{10^{n+m+1}} + \cdots + \frac{b_1}{10^{n+2m}} \right) + \cdots \\ &= a_0 + \sum_{k=1}^{n} \frac{a_k}{10^k} + \frac{1}{10^n} \sum_{k=1}^{m} \frac{b_k}{10^k} + \frac{1}{10^{n+m}} \sum_{k=1}^{m} \frac{b_k}{10^k} + \cdots \\ &= a_0 + \sum_{k=1}^{n} \frac{a_k}{10^k} + \left(\sum_{k=1}^{m} \frac{b_k}{10^k} \right) \left(\frac{1}{10^n} + \frac{1}{10^{n+m}} + \frac{1}{10^{n+2m}} + \cdots \right) \\ &= a_0 + \sum_{k=1}^{n} \frac{a_k}{10^k} + \left(\sum_{k=1}^{m} \frac{b_k}{10^k} \right) \cdot \frac{1}{10^n} \left(1 + \frac{1}{10^m} + \frac{1}{10^{2m}} + \cdots \right) \\ &= a_0 + \sum_{k=1}^{n} \frac{a_k}{10^k} + \left(\sum_{k=1}^{m} \frac{b_k}{10^k} \right) \cdot \frac{1}{10^n} \cdot \frac{10^m}{10^m - 1}. \end{aligned}$$

The right hand side is a finite sum of rational numbers and hence x is a rational number. ☐

3 Binary and Ternary Expansions and the Cantor Set

We shall be brief. Given any point $p \in \mathbb{N}$, $p \geq 2$, we can establish results similar to the decimal forms in the last section for a series of the form

$$a_0 + \frac{a_1}{p} + \frac{a_2}{p^2} + \cdots + \frac{a_n}{p^n} + \cdots$$

where $a_n \in \{0, 1, \ldots, p-1\}$ for each $n \in \mathbb{N}$. The real number α (which is the sum of the above series) is denoted by $a_0.a_1a_2 \ldots a_n \ldots$. The series or this form is called the p-adic expansion of the real number α.

We are mostly interested when $p = 10, 2, 3$. When $p = 2$, (respectively $p = 3$) the expansion is called *binary* (respectively *ternary*). We shall explain the ternary expansion. Let $x \in [0, 1]$. Then we want to write $x = 0.a_1a_2 \ldots a_n \ldots$ (ternary). The digits are defined by locating the point x in a sequence of intervals, the length of which go down by a factor of three each time:

$$\begin{aligned} a_1 &= \sup \left\{ k \in \mathbb{Z} \;\middle|\; \frac{k}{3} \leq x \right\} \\ a_2 &= \sup \left\{ k \in \mathbb{Z} \;\middle|\; \frac{a_1}{3} + \frac{k}{3^2} \leq x \right\} \\ &\vdots \\ a_n &= \sup \left\{ k \in \mathbb{Z} \;\middle|\; \frac{a_1}{3} + \cdots + \frac{a_{n-1}}{3^{n-1}} + \frac{k}{3^n} \leq x \right\} \end{aligned}$$

For instance, if $x = \frac{10}{27}$, then $x = 0.101 = 0.100222\ldots$ (ternary). Note that $a_k \in \{0, 1, 2\}$.

With this brief introduction, we are ready to define Cantor ternary set. We shall give analytical definition first and explain the geometric construction later.

Consider the interval $[0, 1]$, represented in ternary form: for each $x \in [0, 1]$, $x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}$, where $a_k \in \{0, 1, 2\}$ for all $k \in \mathbb{N}$. The Cantor (ternary) set K consists of all those $x \in [0, 1]$ which have a ternary representation $\sum_{k=1}^{\infty} \frac{a_k}{3^k}$, where $a_k \in \{0, 2\}$ for all $k \in \mathbb{N}$.

The geometric construction behind K is as follows:

Take $[0, 1]$ and delete the open middle third interval $(\frac{1}{3}, \frac{2}{3})$. This deletes numbers with 1 in the first ternary place. (The numbers $\frac{1}{3}$ and 1 are not deleted as they have ternary expansions $0.0222\ldots$ and $0.222\ldots$ respectively).

Take the two remaining closed intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ and delete the open middle thirds of these intervals. This deletes numbers with 1 in the second ternary place. (The numbers $\frac{1}{9}$ and $\frac{7}{9}$ are not deleted, because they have ternary expansions $0.00\overline{2}$ and $0.20\overline{2}$ respectively.)

We continue this process ad infinitum. What remains is the Cantor set. As K was obtained by removing an open set U which is the union of a sequence of open intervals $(\frac{1}{3}, \frac{2}{3}), (\frac{1}{9}, \frac{2}{9}), (\frac{7}{9}, \frac{8}{9}), \ldots$, K is closed.

We also note that the sum of the lengths of the disjoint open intervals above is 1:

$$\frac{1}{3} + 2\frac{1}{3^2} + 2^2\frac{1}{3^3} + \cdots + 2^n\frac{1}{3^{n+1}} + \cdots = 1.$$

Thus K has "no length". However K has uncountably many points, more than the obvious points, viz., the end points of the deleted intervals.

This follows from the ternary expansion and the fact that the set of functions from $\mathbb{N}$ to $\{0, 2\}$ has the cardinality of $\mathbb{R}$. More specifically, consider the map $f : K \to [0, 1]$ given by

$$f(\sum_{k=1}^{\infty} \frac{a_k}{3^k}) = \sum_{k=1}^{\infty} \frac{a_k}{2} 2^{-k}.$$

Then f maps K onto $[0, 1]$. However f is not one-to-one. f is called the *Cantor-Lebesgue function*.

Remark 8. Note that at the n^{th} stage of the construction we have a closed set F_n which is the union of 2^n closed intervals of the form $[\frac{k}{3^n}, \frac{k+1}{3^n}]$ for specific k's. For example

$$\begin{aligned} F_1 &= [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \\ F_2 &= [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \end{aligned}$$

Thus K is $\cap F_n$. From this it follows that K contains no non-empty open interval. For, if $(a, b) \subset F_n$ for each n, then $|a - b| \leq \frac{1}{3^n}$ for all n and hence $a = b$ and hence $(a, b) = \emptyset$.

In particular, no connected subset of K can have more than one point.

Let $x \in K$. Then $x \in F_k$ for each k and is therefore a cluster point of the end points of the intervals in F_k. Thus K is a *perfect* set — a set each of whose points is its cluster point.

The Cantor-Lebesgue function admits the following geometric interpretation:

If $U_k := [0,1] \setminus F_k$, then U_k consists of $2^k - 1$ intervals I_j^k (ordered from left to right) removed in the first k stages of the construction of the Cantor set.

Let f_k be the continuous function on $[0,1]$ which satisfies $f_k(0) = 0$, $f_k(1) = 1$, $f_k(x) = j2^{-k}$ on I_j^k, $0 \leq j \leq 2^k - 1$, and which is linear on each interval of F_k. The graphs of f_1 and f_2 are shown below.

By construction, each f_k is monotone increasing $f_{k+1} = f_k$ on I_j^k, $1 \leq j \leq 2^k - 1$ and $|f_k - f_{k+1}| \leq 2^{-k}$. Hence $\sum(f_k - f_{k+1})$ converges uniformly on $[0,1]$ and therefore (f_k) converges uniformly to a continuous function f on $[0,1]$. Then $f(0) = 0$, $f(1) = 1$, f is monotone increasing and continuous on $[0,1]$.

There is another description of f worth noting. Let U be the union of the deleted open intervals. We define f on U by

$$\begin{aligned} f &= \frac{1}{2} \text{ on } (\frac{1}{3}, \frac{2}{3}) \\ f &= \frac{1}{4} \text{ on } (\frac{1}{9}, \frac{2}{9}) \\ f &= \frac{3}{4} \text{ on } (\frac{7}{9}, \frac{8}{9}) \\ f &= \frac{1}{8} \text{ on } (\frac{1}{27}, \frac{2}{27}) \\ f &= \frac{3}{8} \text{ on } (\frac{7}{27}, \frac{8}{27}) \\ &\vdots \qquad\quad \vdots \end{aligned}$$

Clearly, $|f(x) - f(y)| < 2^{-n}$ if $|x - y| < 3^n$ for $x, y \in U$. Then f is uniformly continuous on U and hence has a unique continuous extension $g : \overline{U} = [0,1] \to [0,1]$. Notice that the graph of f is flat practically everywhere, but g still manages to climb from 0 to 1 continuously.

Given $x \in K$, we can write $x = \sum_{k=1}^{\infty} \frac{2}{3^k}\varepsilon_k$, where $\varepsilon_k \in \{0,1\}$. By continuity of f, and the geometric construction of K and f,

$$f(x) = \lim_{n\to\infty} f(\sum_{k=1}^{n} \frac{2}{3^k}\varepsilon_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\varepsilon_k}{2^k} = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^k}.$$

In fact we can write f explicitly in terms of ternary expansion and binary expansion. Let $x = 0.x_1x_2\cdots x_n\cdots$ (ternary). If $x \in K$, so that $x_n \in \{0,2\}$ for all $n \in \mathbb{N}$, we let $f(x) = 0.\frac{x_1}{2}\cdot\frac{x_2}{2}\cdots\frac{x_n}{2}\cdots$ (binary).

If $x \notin K$, let N be the least integer such that $x_N = 1$. We then define $f(x) = 0.b_1b_2\cdots b_Nb_{N+1}\cdots$, where

$$b_i = \begin{cases} \frac{x_i}{2} & 1 \leq i \leq N-1 \\ 1 & i = N \\ 0 & i > N \end{cases}$$

To see that f coincides with our earlier construction, observe that if $x = 0.x_1x_2\cdots x_n\cdots$ (ternary) is such that $x_1 = 1$, then $x \in (\frac{1}{3}, \frac{2}{3})$ so that $f(x) = \frac{1}{2}$ on $(\frac{1}{3}, \frac{2}{3})$. If $x_1 \in \{0,2\}$

and $x_2 = 1$, then x lies either in $(\frac{1}{9}, \frac{2}{9})$ or $(\frac{7}{9}, \frac{8}{9})$ according as $x_1 = 0$ or $x_1 = 2$. Thus in the former case, $f(x) = 0.01 = \frac{1}{2^2} = \frac{1}{4}$ (binary) and in the latter case, $f(x) = 0.11 = \frac{1}{2} + \frac{1}{2^2} = \frac{3}{4}$. This way, we see that this new function f coincides with the one defined on U, the open set consisting of deleted middle third open intervals.

f is obviously continuous on the open set U (why?). These two constructions coincide on the dense set U and hence they coincide on all of $[0, 1]$.