Outline of the Differential Geometry Course (2006-2007)

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1. Review of Euclidean spaces: Dot product on $\mathbb{R}^n$ and its properties.

2. Inner product spaces; new inner products on $\mathbb{R}^2$; specific example was $(v, w) \mapsto {}^t wAv$ where $A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$. Here $v, w \in \mathbb{R}^2$ are thought of as 2×1 matrices (or as column vectors).

3. The principle behind the choice of the matrix was explained. HW: Think of another matrix A and show that we do get an inner product.

4. Statement and proof of Cauchy-Schwarz inequality. Need only consider when the vectors are nonzero and hence unit vectors v, w. Expand $\langle v \pm w, v \pm w\rangle$ and arrive at the inequality. Equality case was also dealt with.

5. **Exercise:** The proof indicated for CS inequality can be adapted for (hermitian) inner product spaces over $\mathbb{C}$. Do this.

6. Orthonormal basis: definition.

Items 1–6 were done on 26-6-2006.

7. If $x \in V$, a vector in an inner product space and $\{v_i : 1 \leq i \leq n\}$ is an O.N. basis, then $x = \sum_{k=1}^n x_k v_k$ where $x_k = \langle x, v_k\rangle$ for $1 \leq k \leq n$.

8. Any linear map $f \colon \mathbb{R}^n \to \mathbb{R}$ is of the form $f(x) = x.u$ where $u = (f(e_1), \dots, f(e_n))$.

9. The nullity of any nonzero linear functional f is $n - 1$ so that $\mathbb{R}^n$ is the orthogonal direct sum of $W = \ker f$ and $\mathbb{R}u$.

10. Orthogonal (linear) maps: equivalent conditions: (i) preserving inner product; (ii) preserving the norm and (iii) taking an O.N. basis to another O.N. basis.

11. Examples of orthogonal maps in $\mathbb{R}^2$: the scalar map λI is orthogonal iff $\lambda = \pm 1$; rotations and reflections; we also dealt with reflection R_H along a hyperplane H through the origin, where $H = \ker f = u^\perp$, for some nonzero u. If u is a unit vector, then any $x \in \mathbb{R}^n = \ker f \oplus \mathbb{R}u$ is written as $x = y + tu$ and $R_H(x)y - tu$. R_H is orthogonal and has determinant -1.

12. Distance preserving maps between metric spaces and isometries. Any distance preserving map is 1-1. Compositions of distance preserving maps are distance preserving.

13. Any orthogonal map and translations are distance preserving maps of $\mathbb{R}^n$ (with standard dot product, norm and the corresponding metric).

14. *Any distance preserving map* $f\colon \mathbb{R}^n \to \mathbb{R}^n$ *with* $f(0) = 0$ *is of the form* $x \mapsto Tx$ *where* T *is an orthogonal map.*

    - Step 1. $\| f(x)\| = \|x\|$ for $x \in \mathbb{R}^n$.
    - Step 2. $\langle f(x), f(y)\rangle = \langle x, y\rangle$ for $x, y \in \mathbb{R}^n$.

Items 7–14 were done on 27-6-2006.

15. Step 3. As a corollary of Step 2, f carries an O.N. basis to an O.N. basis. Using this information, we show that $f(x) = \sum_{i=1}^{n} \langle x, e_i\rangle f(e_i)$ if $x = \sum_i \langle x, e_i\rangle e_i$. Hence f is linear. This completes the proof of the theorem in Item 14.

16. Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be distance preserving. Then there exists an orthogonal map A and $v \in \mathbb{R}^n$ such that $f(x) = Ax + v$ for $x \in \mathbb{R}^n$. Hint: Take $v = f(0)$ and consider $g(x) = f(x) - v$. Then g satisfies the hypothesis of Item 14. In particular, any distance preserving map of $\mathbb{R}^n$ is onto and hence is an isometry.

Only Items 15 and 16 were done on 4-7-2006 due to rains!

17. (Hyper-)Planes H in $\mathbb{R}^{n+1}$ were defined in two ways:

    (a) As $f^{-1}(d)$ where $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ is a nonzero linear functional. This is the same as saying $H = \{x \in \mathbb{R}^{n+1} : x \cdot a = d\}$ for a nonzero vector $a \in \mathbb{R}^{n+1}$.

    (b) As cosets $y + W$ of a vector subspace $W \leq \mathbb{R}^{n+1}$ of dimension n.

18. Using the orthogonal decomposition $\mathbb{R}^{n+1} = W \oplus \mathbb{R}u$ for some unit vector u, we found a formula for $d(Q, W)$ for $Q \in \mathbb{R}^{n+1}$:

$$d(Q, W) = |t| \text{ if } Q = x + tu, x \in W, t \in \mathbb{R}.$$

As a corollary, if $H = y + W$ if any hyperplane, then

$$d(Q, H) = |f(Q) - d|\,,$$

where $f(y) = d$. In case a is not a unit vector, we arrive at

$$d(Q, H) = \frac{\left|\sum_{i=1}^{n+1} a_i y_i - d\right|}{\|a\|},$$

where $Q = (y_1, \dots, y_{n+1})$ and $H = f^{-1}(d)$ with $f(x) = x \cdot a$.

Reference for the all the items so far is Chapter 5 (especially Sections 5.6–5.9) of *Linear Algebra — A Geometric Approach* by S. Kumaresan.

19. The planes arise in differential geometry as the tangent planes: if $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, then the graph $G(f) := \{(x, f(x)) : x \in U\}$ can be thought of as a (hyper)surface in $\mathbb{R}^{n+1}$. The tangent (hyper)plane at the point $(a, f(a)) \in G(f)$ is given by

$$x_{n+1} = f(a) + Df(a)(x - a).$$

When $n = 2$, and the point is $(a, b) \in U \subset \mathbb{R}^2$, the derivative $DF(a, b)(h, k) = \frac{\partial f}{\partial x}(a, b)h + \frac{\partial f}{\partial y}(a, b)k$. In this case the tangent plane to the graph is given by

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).$$

Note that the point $(a, b, f(a, b))$ lies on the graph as well as on the plane. In fact, when we say that f is differentiable at (a, b), the condition means the existence of tangent plane to the graph of the function f at the point.

20. During the discussion of Item 19, we recalled a precise version of a well-known result: *If $f\colon U \to \mathbb{R}$ is differentiable at p, then the directional derivatives $D_v f(p)$ exist and we have*

$$\boxed{D_v f(p) = Df(p)(v).}$$

21. The vector a of Item 17 or u of Item 18 are called the *normals* to the plane. The normal to the tangent plane of Item 19 is $(\frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b), -1)$. And, the tangent plane is $(x, y, z) \cdot (\frac{\partial f}{\partial x}(a, b), \frac{\partial f}{\partial y}(a, b), -1) = -f(a, b) + a\frac{\partial f}{\partial x}(a, b) + b\frac{\partial f}{\partial y}(a, b)$!

> Items 17–20 were done on 6-7-2006. The reference for Items 19–21 is my article "A Conceptual Introduction to Several Variable Calculus" which is available in MTTS notes and from me for photo-copying.

The next few items deal with the notion of tangents which is of utmost importance in differential geometry.

22. A line in a real vector space V: If $u, v \in V$, then the line joining u, v is defined by

$$\ell(u, v) := \{x \in V : x = u + t(v - u), \text{ for some } t \in \mathbb{R}\}.$$

It is seen that this gives rise to the equation of a line joining two points in $\mathbb{R}^2$ or $\mathbb{R}^3$.

23. Equivalently, a line through $p \in V$ and in the direction of $0 \neq v \in V$ is given by

$$\ell(p; v) := \{x \in \mathbb{R}^n : x = p + tv \text{ for some } t \in \mathbb{R}\}.$$

v is called the direction vector of $\ell(p, v)$ and corresponds to the 'slope' of ℓ. Reference to lines in a real vector space is Section 3.1 of my book on Linear Algebra.

24. Let $c\colon (a, b) \to U \subset \mathbb{R}^n$ be a (continuously differentiable) curve. If we write $c(t) = (x_1(t), \ldots, x_n(t))$, then $c'(t) = (x_1'(t), \ldots, x_n'(t))$ is called the tangent (or the velocity) vector to the curve c at t.

25. Why is $c'(t)$ called the tangent vector? If c is a parametrization of a standard conic section in $\mathbb{R}^2$, then the tangent line at $c(t)$ is the line through $c(t)$ in the direction of $c'(t)$. Details for circles and ellipses were done in the class and those for hyperbolas and parabolas were left as home-work. Reference for the last two items is Section 5.4 of *An Expedition to Geometry* by Kumaresan and Santhanam.

26. If $S \subset \mathbb{R}^n$ and $p \in S$, we denote by T_pS the tangent space at p to S and define it as the collection of all tangent vectors $c'(0)$ where $c\colon (-\varepsilon, \varepsilon) \to S$ is a smooth (as a map from $(-\varepsilon, \varepsilon)$ to $\mathbb{R}^n$) curve with $c(0) = p$:

$$T_pS := \{v \in \mathbb{R}^n : \exists c\colon (-\varepsilon, \varepsilon) \to S \text{ with } c(0) = p \text{ and } v = c'(0)\}$$

27. As $c(t) = p$ for all t has 0 as the tangent vector, $T_pS \neq \emptyset$. Also, if $v \in T_p$, then $\lambda v \in T_pS$ for any $\lambda \in \mathbb{R}$. Thus T_pS is a subset of $\mathbb{R}^n$ which is nonempty and closed under scalar multiplication. Whether it contains nonzero vectors or whether it is closed under (vector) addition (so that it becomes a vector space) depends on some geometric properties of S. We look at some special cases below which are very important for differential geometry and for which the tangent spaces are vector spaces.

28. If $S = U$ is an open subset of $\mathbb{R}^n$, then $T_p(S) = \mathbb{R}^n$.

29. If S is a the vector subspace $W := \{x \in \mathbb{R}^n : x \cdot a = 0\}$ for some nonzero $a \in \mathbb{R}^n$, then $T_WS = W$. If $v \in T_pS$, then $v \in W$ with a corresponding curve c, then $c(t) \cdot a = 0$ for all t. Differentiating this equation we get $v \in W$. (We also saw why W was closed. That it was not open was seen in two ways.).

Items 22–29 were done on 11-7-2006. The topic of tangents was chosen in place of DE by a toss of a coin!

30. More generally, if $H := w + W$ is a plane, then $H = p + W$ for any $p \in H$ and $T_pH = W$.

31. Let a plane W passing through the origin be given by $x \cdot a = 0$. Then the equation for the plane $p + W$, the translate of W by p, is given by $x \cdot a = p \cdot a$. (This reminds us of the high-school geometry formula: the equation of the line through (x_0, y_0) parallel to the line $ax + by = 0$ is $ax + by = ax_0 + by_0$.)

32. Let $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}$ be smooth. Let S be the "surface" in $\mathbb{R}^{n+1}$ defined as the graph of f: $S := \{(x, f(x)) : x \in U\}$. If γ is a curve in S, and if we write $\gamma(t) = (x(t), f(x(t))$, then $c(t) := x(t)$ is a curve in U. This sets up a 1-1 correspondence between curves in S and curves in U. If $n = 2$, so that

$$\gamma(t) = (x(t), y(t), z(t)) = (x(t), y(t), f(x(t), y(t))),$$

then

$$\begin{aligned} \gamma'(t) &= \left(x'(t), y'(t), \frac{\partial f}{\partial x}x'(t) + \frac{\partial f}{\partial y}y'(t)\right) \\ &= x'(t)\left(1, 0, \frac{\partial f}{\partial x}\right) + y'(t)\left(0, 1, \frac{\partial f}{\partial y}\right) \\ &= x'(t)\partial_x + y'(t)\partial_y, \text{ say.} \end{aligned}$$

∂_x and ∂_y are the tangent vectors to S that correspond to the curves $t \mapsto (a+t, b, f(a+t, b))$ and $t \mapsto (a, b+t, f(a, b+t))$. Note that these curves are obtained as the intersections of the planes $y = b$ and $x = a$ with the surface S. These tangent vectors will be referred to as the coordinate tangent vectors.

More generally, if we consider $c : t \mapsto (a + tu, b + tv, f(a + t, b + t))$, then $c(0) = p$ and $c'(0) = u\partial_x + v\partial_y$.

The general n-dimensional case was also rushed through!

While computing $\gamma'(t)$, we recalled the chain rule and used it to show that $\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x} x'(t) + \frac{\partial f}{\partial y} y'(t)$.

33. Thus any tangent vector $v \in T_pS$ is a linear combination of the coordinate tangent vectors ∂_x and ∂_y. Hence $T_pS = \text{span}\,\{\partial_x, \partial_y\}$.

34. Also, note that

$$\partial_x \times \partial_y = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right)$$

the 'normal' to the tangent space T_pS. Hence the tangent space T_pS is given by $\{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \cdot \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) = 0\}$.

35. The tangent plane is by definition the plane through $(a, b, f(a, b)) \in S$ parallel to T_pS, that is , the plane $p + T_pS$ and is given by

$$(x, y, z) \cdot \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right) = (a, b, f(a, b)) \cdot \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1\right).$$

Thus the equation of the tangent plane is

$$z = (x - a)\frac{\partial f}{\partial x} + (y - b)\frac{\partial f}{\partial y} + f(a, b).$$

36. The geometric meaning of the existence of the derivative of f at (a, b) is the existence of a tangent plane which 'approximates' the graph of f at $(a, b, f(a, b))$. The rigorous definition quantifies what is meant by approximation. (This was already hinted in Item 19.) For pictures that illustrate Items 32–36, refer to my article quoted in the box below Item 21. I also offered to show them on my computer screen!.

37. Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function. Assume that 0 is a regular value of f, that is, 0 is a value of f and that if $f(p) = 0$ then $\text{grad}\, f(p) \neq 0$. Let $S := f^{-1}(0)$, called the level set of f at the level 0. We wish to show that $T_pS = (\text{grad}\, f(p))^\perp$. As earlier, it is easy to show that $T_pS \subset \text{grad}\, f(p)^\perp$.

38. During the discussion of the last few items, we also had an intuitive discussion on the concept of dimensions.

Items 30–38 were done on 13-7-2006. Extra classes on Saturdays were announced. You may refer to the notes being written by Jizelle/Rashmi/Simi.

39. We wanted to show that $T_pS = \text{grad}\, f(p)^\perp$ for S as in Item 37. It is easy, provided that one knows the implicit function theorem in a proper way.

40. We looked at some of the standard examples of level sets such as a plane, sphere, cylinder and a saddle surface (given by $z = xy$). In all the cases, we found that if $\frac{\partial f}{\partial x_i}(p) \neq 0$, then we can express x_i as a function of other coordinates in an open (in the subspace topology) set containing p.

41. We recalled the implicit function theorem for functions $f\colon \mathbb{R}^{n+1} \to \mathbb{R}$ with motivations. The main emphasis was to exhibit any level set in $\mathbb{R}^{n+1}$ locally as a graph of a function defined on an open set in $\mathbb{R}^n$.

    Let $f\colon \Omega \subset \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth (continuously differentiable, at least) function. Let $(a_1, \ldots, a_n, b)$ be such that $f(a,b) = 0$ and $\frac{\partial f}{\partial x_{n+1}}(a,b) \neq 0$. Then there exists an open set $U \subset \mathbb{R}^{n+1}$ with $(a,b) \in U \subset \Omega$, an open set $V \ni a$ in $\mathbb{R}^n$ and a smooth function $g\colon V \to \mathbb{R}$ such that $g(a) = b$ and $U \cap S = \{(x, g(x)) : x \in V\}$. We also have

    $$\operatorname{grad} g(x) = -\left(\frac{\partial f}{\partial x_{n+1}}\right)^{-1}\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right). \tag{1}$$

    Note that V is homeomorphic to $U \cap S$. For, if $f\colon X \to Y$ is continuous, then the graph of f as a subset of $X \times Y$ with the subspace topology is homeomorphic to X.

    Ex: We established (1) when $n = 1$. Prove the general version, as it is a good exercise in the application of the chain rule.

42. While arriving at the precise statement of the implicit function theorem as in Item 41, we looked at the examples in Item 40 (especially the plane and a circle) to derive inspiration.

43. We now come back to the proof of $T_pS = \operatorname{grad} f(p)^{\perp}$ for any level set S. Assume that the $(n+1)$-th partial derivative of f at p is nonzero. Let U, V, g be as in the statement of the implicit function theorem. Let S_1 be the graph of g: $S_1 := \{(x, g(x)) : x \in V\}$. We claim that any $v \in T_pS_1$ also lies in T_pS. This is clear since if c is curve in S_1 with the initial data $c(0) = p$ and $c'(0) = v$, then c is also a curve in S, (as $S_1 = U \cap S$) with the same initial data. From Item 33, we know that T_pS_1 is n-dimensional. Hence T_pS contains the n-dimensional vector space T_pS_1 and is contained in the n-dimensional space $\operatorname{grad} f(p)^{\perp}$ by Item 37. We conclude that $T_pS = \operatorname{grad} f(p)^{\perp}$.

44. A different proof (which, for a given $v \perp \operatorname{grad} f(p)$, exhibits a curve c with $c(0) = p$ and $c'(0) = v$) is given in my article "Lagrange Multipliers — A Geometric Treatment". It is available in the MTTS notes. We may look at the proof later.

45. We recalled that the notion of continuity and differentiability of a function at a point are 'local' concepts. This means that to check whether a function is continuous (or differentiable) at a point, we need to know the value of the function *only* on an open set containing the point under discussion. This observation was used in the last item.

46. We interpreted the domain of the derivative $Df(p)$ of a differentiable function on an open set $U \subset \mathbb{R}^n$ as T_pU: to know $Df(p)$ it is enough to know $Df(p)(v)$. The latter is the directional derivative $D_vf(p)$ and to compute it, we can use any curve γ such that $\gamma(0) = p$ and $\gamma'(0) = v$. (This follows by a trivial application of the chain rule.) Hence v, which is fed to $Df(p)$, can be thought of a tangent vector to U at p!

47. The interpretation of the domain of the derivative in the last item leads us to the definition of the derivative of a map between two surfaces. In fact, some of you immediately came up with the idea in the class!

Items 39–47 were done on 15-7-2006. You are advised to write your own notes to develop your writing skills and refer to the notes of Jizelle/Rashmi/Simi only if required.

48. As a second instance where the knowledge of tangent spaces arises, we took up the task of ferreting out the geometric idea behind the method of Lagrange multiplier.

49. To achieve the goal mentioned in the last item, we looked at the following easy problems of constrained maxima/minima in which it was easy to discern the underlying geometric phenomenon. We are given the constraint $f(x, y) = 0$ and are required to find the extrema of the function g subject to the constraint f. We interpreted this problem as that of finding the extrema of g as a function on the level set $S = f^{-1}(0)$.

    (a) $f(x, y) = x^2 + y^2 - 1$ and $g(x, y) = x$.

    (b) $f(x, y) = xy - 1$ and $g(x, y) = x^2 + y^2$.

    (c) $f(x, y) = (x/a)^2 + (y/b)^2 - 1$ and $g(x, y) = x^2 + y^2$.

    (d) Find the rectangle whose are is a maximum if its perimeter is of fixed length L. Here $f(x, y) = 2(x + y) - L = 0$ and $g(x, y) = xy$.

    What we observed was that if $p \in S$ is a point at which g has an extremum, say $c = g(p)$, then the level sets S and $g^{-1}(c)$ meet at p "tangentially".

50. Since we know the tangent spaces of level sets, our observation can be formulated in terms of the normals to the level sets S and $g^{-1}(c)$ as follows: if $p \in S$ is a point at which g has an extremum, say $c = g(p)$, then the gradients of f and g at p are parallel. Since $\operatorname{grad} f(p)$ is nonzero by assumption, this means that $\operatorname{grad} g(p) = \lambda \operatorname{grad} f(p)$ for some $\lambda \in \mathbb{R}$. Thus we arrived at Lagrange multiplier method.

51. **Theorem:** *Let $f\colon U \to \mathbb{R}$ be a smooth function such that 0 is a regular value, that is, 0 is a value of f and at each point $p \in S := f^{-1}(0)$, we have $\operatorname{grad} f(p) \neq 0$. Let $g\colon U \to \mathbb{R}$ be a smooth function. Assume that $p \in S$ is a point of local maximum/minimum for g on S. Then there exists $\lambda \in \mathbb{R}$ such that $\operatorname{grad} g(p) = \lambda \operatorname{grad} f(p)$.*

    The proof was easy. Let $v \in T_pS$. Let c be a curve with initial data $c(0) = p$ and $c'(0) = v$. Then the function $h(t) := g \circ c(t)$ has a local maximum at $t = 0$ and hence $h'(0) = 0$. But $h'(0) = Dg(p)(c'(0)) = \operatorname{grad} g(p) \cdot c'(0)$. Hence $\operatorname{grad} g(p) \perp T_pS$. Since T_pS is $n-1$ dimensional, $\mathbb{R}^n = TpS \oplus \mathbb{R} \operatorname{grad} f(p)$. It follows that $\operatorname{grad} g(p) \in \mathbb{R} \operatorname{grad} f(p)$. □

52. We did the following problems now 'rigorously'. Find the extrema of g subject to the constraint $f = 0$ where

    (a) $f(x, y) = x^2 + y^2 - 1$ and $g(x, y) = x$.

    (b) $f(x, y) = x + y - L$ and $g(x, y) = xy$.

    (c) $f(x) = x \cdot a$ and $g(x) := \|x - p\|^2$. ($a, p \in \mathbb{R}^n$ are fixed vectors.)

53. We also proved the existence of a real eigenvalue for a symmetric real matrix A. Here $f(x) = (x \cdot x) - 1$ and $g(x) = Ax \cdot x$. We found that $\operatorname{grad} f(p) = 2x$ and $\operatorname{grad} g(x) = 2Ax$.

    The references for the items 48–53 are my articles (i) Implicit function theorem and (ii) Lagrange Multipliers–A Geometric Approach. Both are available in MTTS notes. Pictures for some of the problems may be found in the first article as well as in the electronic version of the MTTS notes.

Items 48–53 were done on 19-7-2006.

54. I asked you to write down the statement of the implicit function theorem in the last class (on 19/7/2006) as well as on 20/7/2006. You know how much you succeeded!

55. We recalled the statement of the inverse function theorem (with no motivation at all, as I was upset with you!)

56. We deduced the implicit function theorem from the inverse function theorem (with no pictures, but with full justification).

57. Later I explained the proof with motivations and picture.

Items 54–57 were done on 20-7-2006. Reference for this material is any standard book on analysis such as the ones by Apostol or Rudin.

58. A (nonempty) subset $S \subset \mathbb{R}^n$ is called a *surface* in $\mathbb{R}^n$ if for each $p \in S$, there exists an open neighbourhood U of p in $\mathbb{R}^n$, an open set $V \subset \mathbb{R}^2$ and a homeomorphism $f \colon V \to U \cap S$.

59. Examples of surfaces:

    (a) A nonempty open set $V \subset \mathbb{R}^2$ is a surface in $\mathbb{R}^2$.

    (b) More generally, if $W \subset \mathbb{R}^n$ is a two dimensional vector subspace and V is an open set in W (with the subspace topology on W), then V is a surface in $\mathbb{R}^n$.

    (c) A typical example is the sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. If we let $U_i^{\pm} := \{(x_1, x_2, x_3) \in S : \pm x_i > 0\}$, $i = 1, 2, 3$, then each of these six sets is open in S and is homeomorphic to the open unit disk in $\mathbb{R}^2$. Also, any $p \in S$ will lie in at least one of these open sets.

    (d) Let $U \subset \mathbb{R}^2$ be open. Let $f \colon U \to \mathbb{R}^k$ be a (continuous, or better still smooth) map. Then the graph $G(f) := \{x, f(x)) \in \mathbb{R}^2 \times \mathbb{R}^k = \mathbb{R}^{k+2}\}$ is a surface in $\mathbb{R}^{k+2}$.

    (e) Let $f \colon \mathbb{R}^3 \to \mathbb{R}$ be a smooth map with 0 as a regular value. Then the level set $S := f^{-1}(0)$ is 'locally' a graph of a function defined on an open set in $\mathbb{R}^2$. (This was seen earlier in Item 43 and also in the notes "T_pS for two classes of surfaces.") Hence S is a surface in $\mathbb{R}^3$.

60. We can mimic the definition of a surface to define k dimensional objects in $\mathbb{R}^n$. They are the ones where each point has a neighbourhood which is homeomorphic to an open set in $\mathbb{R}^k$. They are called k dimensional (topological) submanifolds in $\mathbb{R}^n$.

61. We then defined a smooth surface in $\mathbb{R}^n$ as follows: A nonempty subset $S \subset \mathbb{R}^n$ is a smooth surface if it satisfies the following conditions:

(i) For each $p \in S$, there exists an open set $V \in \mathbb{R}^2$ and an open neighbourhood U of p in S and a homeomorphism $f \colon V \to U$.

(ii) Each of the maps f in (i) is a smooth map when considered as a map from V to $\mathbb{R}^n$.

(iii) The Jacobian matrix $Df(u,v)$ has rank 2 at each point $(u,v) \in V$.

62. We interpreted the condition (iii) in the definition of a surface as follows: Consider the u-coordinate (straight line) curve through a point $(u,v) \in V$ given by $t \mapsto ((u+t,v)$ and its image $c_u \colon t \mapsto (x_1(u+t,v), x_2(u+t,v), \dots, x_n(u+t,v))$. Its tangent vector to S at (u,v) is $(\frac{\partial x_1}{\partial u}(u,v), \dots, \frac{\partial x_n}{\partial u}(u,v))$. Similarly, we can define the v-coordinate curve and its tangent vector. Clearly, they constitute the first and the second columns of the Jacobian matrix of $Df(u,v)$. So (iii) simply means that these two tangent vectors are linearly independent and hence they span a two dimensional tangent space to S at $f(u,v)$.

63. Similarly, one defines a k-dimensional smooth manifold in $\mathbb{R}^n$.

Items 58–63 were done on 22-7-2006. References for most of the topics in DG are: (i) do Carmo: *Differential Geometry of Curves and Surfaces* and (ii) A. Pressley: *Elementary Differential Geometry*. Both the books are available in the student library.

64. **Quiz**

(a) Given an O.N. basis $\{v_i : 1 \le i \le n\}$ of an inner product space V, a vector $v \in V$ and $v = \sum_{i=1}^n c_i v_i$, identify the scalars c_i.

(b) Any linear map $f\colon \mathbb{R}^n \to \mathbb{R}$ is of the form $f(x) = x \cdot a$. Identify the vector a (as an n-tuple).

(c) True or false? For $n \ge 2$, the linear map $(x_1, x_2, \dots, x_n) \mapsto (x_1, 2x_2, \dots, nx_n)$ on $\mathbb{R}^n$ with the standard dot product is orthogonal.

(d) True or false? Any linear map on $\mathbb{R}^n$ which preserves the norm also preserves the dot product.

(e) What are all the distance preserving maps of $\mathbb{R}^n$?

(f) True or false? Any distance preserving map of $\mathbb{R}^n$ is an isometry.

(g) If a plane W in $\mathbb{R}^n$ is given by $x \cdot a = 0$ what is the equation of the plane passing through the point p?

(h) Given two (linearly independent) vectors $x, y \in \mathbb{R}^n$, let $\ell(x, y)$ denote the line joining them. Write it in the form $\ell(x; d)$ as the line through x in the direction of a nonzero vector d.

(i) Define T_pS where $S \subset \mathbb{R}^n$.

(j) Let $S = \{(x, y, f(x, y)) : (x, y) \in U\}$ be the graph of a smooth function on an open set $U \subset \mathbb{R}^2$. Give 2 linearly independent vectors in T_pS and the corresponding curves in S.

(k) Let S be the graph of $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = xy$. Give the equations of the tangent space and the tangent plane to S at $(1, 1, 1) \in S$.

(l) Write down the equation of the plane that appears in the definition of differentiability of a function on $\mathbb{R}^2$.

(m) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be differentiable, $a, v \in \mathbb{R}^n$. What is the relation between $D_v f(a)$ and $Df(a)$?

(n) Let $c\colon \mathbb{R} \to \mathbb{R}^n$ be a curve and $f\colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. What is $\frac{d}{dt}(f \circ c)(t)$?

(o) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and $g\colon \mathbb{R} \to \mathbb{R}$ be smooth. Assume that $f(x, g(x)) = 0$ for all $x \in \mathbb{R}$. What is $g'(x)$?

(p) Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function with 0 as a regular value. Identify (without proof) T_pS where $S = f^{-1}(0)$.

(q) Let S be the sphere $x^2 + y^2 + z^2 = 1$. Find the equations of the tangent space and the tangent plane at the point $(1/\sqrt{2}, 0, 1/\sqrt{2})$.

(r) Consider a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$ with 0 as a regular value. Let $S := f^{-1}(0)$. Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be a smooth function whose restriction to S attains a maximum, say, M at $p \in S$. Then $p \in S_1 := g^{-1}(M)$. What is the relation between the tangent spaces T_pS and T_pS_1?

(s) Show that the graph of $f\colon \mathbb{R}^2 \to \mathbb{R}$ can be considered as a level set of a suitably defined function $g\colon \mathbb{R}^3 \to \mathbb{R}$.

(t) State the inverse function theorem or the implicit function theorem without proof.

You had the Quiz of Item 64 on 24-7-2006. The quiz lasted for about an hour and its solution along with corrections by your neighbours took about another hour! You all welcomed the idea of having such a quiz after any reasonable amount of coverage of topics. Let us hope we shall have them as often as required. Please report to me any corrections of the hand-outs **when** I am in my office so that I could incorporate them in the computer.

65. We recalled the definition of a (smooth) surface, especially Items 61–62.

66. We showed that our examples of surfaces in Item 59 are smooth surfaces.

67. The map $f\colon V \to \mathbb{R}^n$ appearing in the definition of a smooth surface is called a *patch*. We showed that the unit sphere S^2 in $\mathbb{R}^3$ is a smooth surface in many ways, with an infinite number of patches, with six patches, with two patches.

68. No compact surface S can "admit a single patch." The argument was purely topological one using the compactness S and connectedness of $\mathbb{R}^2$.

69. We discussed the spherical polar coordinates; how to arrive at them. Exercise: Adapt the same method to find a patch for the ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ and the spherical polar coordinates for the n-dimensional sphere in $\mathbb{R}^{n+1}$.

70. We defined the surface of a revolution about the z-axis of a curve $f(u) := (x(u), 0, z(u))$, $a < u < b$, in the xz plane. We needed two patches:

$$(u,v) \mapsto (\cos v\, x(u), \sin v\, x(u), z(u)), \text{ where } \begin{cases} (u,v) \in (a,b) \times (0, 2\pi), & \text{first patch,} \\ (u,v) \in (a,b) \times (-\pi, \pi), & \text{second patch.} \end{cases}$$

71. Special cases of surfaces of revolution are: (i) cylinder, (ii) sphere and (iii) the cone given by $x^2 + y^2 - z^2 = 0$ got by revolving the half-line $\{(u, 0, u) : u > 0\}$ about the z-axis.

72. We defined the u and v coordinate curves of a given patch of a surface. We looked at these curves in the special cases of a surface of revolution. In the case of a sphere, we found them to be longitudes and latitudes.

Items 64-72 were done on 27-7-2006. You were also shown the answer-books for checking. You are advised to go through the proof of implicit function theorem as the argument is needed on Saturday's lecture.

73. **Examples of Parametrization of Surfaces**

 (a) Ellipsoid: $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$. A parametrization is

$$\varphi(u,v) := (a \cos u \cos v, b \cos u \sin v, c \sin u)$$

 where the domain of f is $U := (-\pi/2, \pi/2) \times (0, 2\pi)$.

 (b) Hyperboloid of one sheet: $(x/a)^2 + (y/b)^2 - (z/c)^2 = 1$. A parametrization is $\varphi(u,v) := (a \cosh u \cos v, b \cosh u \sin v, c \sinh u)$ where the domain of f is $U := \mathbb{R} \times (0, 2\pi)$.

(c) Hyperboloid of two sheets: $(x/a)^2 - (y/b)^2 - (z/c)^2 = 1$. A parametrization is $\varphi(u,v) := (a\cosh u, b\sinh u\cos v, c\sinh u\sin v)$ where the domain of f is $U := \mathbb{R}\times(0,2\pi)$.

(d) Elliptic paraboloid: $(x/a)^2 + (y/b)^2 = z$. A parametrization is

$$\varphi(u,v) := (v\,a\cos u, v\,b\sin u, v^2), \text{ where } (u,v)\in U := (0,2\pi)\times(0,\infty).$$

(e) Hyperbolic paraboloid: $(x/a)^2 - (y/b)^2 = z$. A parametrization of the portion where $z > 0$ is $\varphi(u,v) := (v\,a\cosh u, v\,b\sinh u, v^2)$ where the domain of f is $U := (0,2\pi)\times(0,\infty)$.

(f) Saddle surface (or hyperbolic paraboloid): Given by $z = xy$ and parametrized by $(u,v)\mapsto(u,v,uv)$. Find the coordinate curves.

(g) Find a parametrization (or patch) for the hyperbolic paraboloid $(x/a)^2-(y/b)^2 = z$ using the linear change of variables $u := \frac{x}{a}+\frac{y}{b}$ and $v := \frac{x}{a}-\frac{y}{b}$. What are the u and v coordinate curves? Answer: $\varphi(u,v) = \left(\frac{a}{2}(u+v), \frac{b}{2}(u-v), uv\right)$. The coordinate curves are straight lines.

(h) Torus: This is a surface of revolution got by rotating a circle in the xz-plane of radius $a > 0$ with centre at $(b,0,0)$ with $b > a$. The parametrization is given by

$$\varphi(u,v) := ((b + a\cos u)\cos v, (b + a\cos u)\sin v, a\sin u)$$

(i) Monkey's saddle: $z = x^3 - 3xy^2$.

(j) Paraboloid: $z = a(x^2+y^2)$. The obvious parametrizations are $(u,v)\mapsto(u,v,a(u^2+v^2))$ and $(r,\theta)\mapsto(r\cos\theta, r\sin\theta, ar^2)$.

(k) Catenoid as a surface of revolution of a catenary: $x := c\cosh(y/c)$ with $c > 0$. The catenoid therefore is parametrized by

$$(u,v)\mapsto(c\cosh(v/c)\cos u, c\cosh(v/c)\sin u, v).$$

(l) Let $f(x,y,z) := x + xy + yz$. Show that 1 is a regular value of f and show that $S := f^{-1}(1)$ is a smooth surface by exhibiting it as a union of two graph surfaces.

(m) There is a single patch for the cylinder in $\mathbb{R}^3$ defined by $x^2 + y^2 = 1$. Consider the annulus $V := \{(u,v)\in\mathbb{R}^2 : 1 < u^2+v^2 < 9\}$. Define

$$\varphi(u,v) := \left(\frac{u}{\sqrt{u^2+v^2}}, \frac{v}{\sqrt{u^2+v^2}}, \tan\left[\left(\sqrt{u^2+v^2}-2\right)\frac{\pi}{2}\right]\right).$$

(n) Mobius band: It is parametrized by

$$(u,v)\mapsto 2(\cos u, \sin u, 0) + v(\sin\frac{u}{2}, \sin\frac{u}{2}, \cos\frac{u}{2}),\ 0 < u < 2\pi,\ -1 < v < 1.$$

For details, refer to Page 211 of my book *A Course in Differential Geometry and Lie Groups*.

Items 73b, 73f and 73l were done in the class. Perhaps, at a later date, I should explain how I arrived at the parametrizations in Items 73m–73n. The rest are left as home-work for you.

74. **Theorem.** *Any (smooth) surface S in $\mathbb{R}^3$ is is locally a graph surface.* That is, for any given $p \in S$, there exists an open set $U \ni p$ in S, an open set $V \subset \mathbb{R}^2$ and a smooth function $f\colon V \to \mathbb{R}$ such that U is the graph of f. □

75. **Theorem.** *Let $S \subset \mathbb{R}^n$ be a surface. Let (φ, V, U) be a patch in S. Let $F\colon W \subset \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map such that $F(W) \subset U$. Then the map $\varphi^{-1} \circ F\colon W \to V$ is smooth.* □

    References for the last two items are: (1) Section 4.7 of Pressley, (2) Proposition 3 on page 63 and Proposition 1 on Page 70 in do Carmo. Also, see the appendices.

    Corollary. Let (V_i, φ_i, U_i), $i = 1, 2$ be two patches with $U_1 \cap U_2 \neq \emptyset$. Then the map $\varphi_2^{-1} \circ \varphi_1\colon \varphi_1^{-1}(U_1 \cap U_2) \to \varphi_2^{-1}(U_1 \cap U_2)$ is a smooth map. □

    One says that the transition maps are smooth. Note that, by symmetry, $\varphi_1^{-1} \circ \varphi_2$ is also smooth so that these maps are diffeomrphisms and inverses of each other.

Items 73-75 were done on 28-7-2006. I announced a take-home exam in which your are asked to write down the statements of the implicit function theorem, the theorems in Items 74–75 and in Item 78 along with their proofs. Last date for submission is 10 August, 2006.

76. Definition of a smooth function from a surface to $\mathbb{R}$.

77. An Observation: Let $S \subset \mathbb{R}^n$ be a surface and Ω an open set in $\mathbb{R}^n$ such that $S \subset \Omega$. The restriction of any smooth function $f\colon \Omega \to \mathbb{R}$ to S is a smooth from S to $\mathbb{R}$. As immediate corollary, we have the following examples of smooth functions on any given smooth surface.

    (a) Let $u \in \mathbb{R}^n$ be a unit vector. Then the *height function* h_u defined by $h_u(x) := x \cdot u$ is smooth on $\mathbb{R}^n$ and hence its restriction to S is smooth.

    (b) Fix $a \in \mathbb{R}^n$ and define $f(x) := \|x - a\|^2$. This is smooth on $\mathbb{R}^n$ and hence its restriction to S is also smooth.

78. **Theorem.** *A function $f\colon S \to \mathbb{R}$ is smooth iff for each $p \in S$, there exists an open set $U \ni p$ in $\mathbb{R}^3$ and a smooth function $g\colon U \to \mathbb{R}$ such that $g = f$ on $S \cap U$.* □

Items 75-77 were done on 31-7-2006. Item 78 will be proved later.

79. We started with a couple of announcements.

    (a) There will be no quiz on 10-8-2006. Instead, it will be on 2-9-2006. I shall be out of the country between 21 August till 1 September. Enjoy my absence!

    (b) You were asked to read the notes "Four Applications of Inverse Function Theorem", especially Theorem 5 before coming to the class on Saturday (12-8-2006).

80. We defined(!) the tangent space T_pS of any surface $S \subset \mathbb{R}^n$. We proved the following theorem.

    Theorem. *Let $S \subset \mathbb{R}^n$ be a surface. Let $p \in S$ and (V, φ, U) be a chart containing p. Then $T_pS = D\varphi(\mathbb{R}^2) = D\varphi(q)\,(T_qV)$ where $\varphi(q) = p$.* □

    What is the significance of this theorem? The left side, namely, T_pS is defined intrinsically whereas the right side $D\varphi(q)(\mathbb{R}^2)$ depends on the parametrization chosen. In classical books, the tangent vector ∂_u is denoted by X_u. The reason for this is that the patch φ is denoted by $\varphi(u,v) = \mathbf{x}(u,v)$ and so X_u is the partial derivative of $\mathbf{x}$ with respect to u. Also, the condition on the rank of $D\varphi(q)$ is formulated as $X_u \times X_v$ is nonzero!

81. We recalled the definition of a smooth function on a surface S in $\mathbb{R}^n$. You arrived at (almost) correct definition of a smooth map F between two surfaces. What was missing in your definition was the continuity of F. We say that F is smooth at p if (i) F is continuous at p and (ii) for any pair of parametrizations (V_i, φ_i, U_i), $i = 1, 2$, of p and $q := F(p)$ respectively with the property that $F(U_1) \subset U_2$, the map

$$\varphi_2^{-1} \circ F \circ \varphi_1 \colon V_1 \to V_2$$

is smooth at $\varphi_1^{-1}(p) \in V_1$.

82. We reinterpreted the domain and the codomain of the derivative $Df(p)$ of a function $f \colon U \subset \mathbb{R}^m \to \mathbb{R}^n$ as T_pU and $T_{f(p)}\mathbb{R}^n$ respectively. For, if $v \in \mathbb{R}^n = T_pU$, and if c is a curve in U such that $c(0) = p$ and $c'(0) = v$, then $\gamma := f \circ c$ is a curve with initial data $\gamma(0) = f(p)$ and $Df(p)(v) = \gamma'(0)$.

83. You arrived at the definition of the derivative of $F \colon S_1 \subset \mathbb{R}^m \to S_2 \subset \mathbb{R}^n$. We proved the following theorem.

Theorem. *Let $F \colon S_1 \to S_2$ be a smooth map. Let $p \in S_1$. If $v \in T_pS_1$, let c be a curve in S_1 with the initial data $c(0) = p$ and $c'(0) = v$. Then define $DF(p)(v) := \gamma'(0)$ where $\gamma(t) := F \circ c(t)$. Then (i) $DF(p)(v)$ is well-defined and (ii) the map $DF(p) \colon T_pS_1 \to T_{F(p)}S_2$ is linear.* □

The idea of the proof was to write c and γ in terms of local coordinates on U_1 and U_2 as follows.

$$\begin{aligned} c : &= t \mapsto ((u(t), v(t)) = \varphi_1^{-1} \circ c(t) \mapsto (x_1(u(t), v(t)), \dots, x_n(u(t), v(t))) \\ \gamma : &= t \mapsto c(t) \mapsto F(c(t)) := (y_1(x(u(t), v(t)))), \dots y_n(x(u(t), v(t)))), \end{aligned}$$

where $x(u(t), v(t)) := (x_1(u(t), v(t)), \dots, x_n(u(t), v(t)))$. Then $v := c'(0) = u'(0)\partial_u + v'(0)\partial_v$ and we have

$$DF(p)(v) := \gamma'(0) = \left(\frac{\partial y_i}{\partial x_j}\right)(c'(0)) = u'(0)\left(\frac{\partial y_i}{\partial x_j}\right)\partial_u + v'(0)\left(\frac{\partial y_i}{\partial x_j}\right)\partial_v.$$

Observe that the Jacobian $\left(\frac{\partial y_i}{\partial x_j}\right)$ is an $n \times m$ matrix while $\partial_u = \left(\frac{\partial x_1}{\partial u}, \dots, \frac{\partial x_1}{\partial u}\right)^t$ is an $m \times 1$ column vector so the resulting matrix is a column vector of size n. The displayed formula shows that $DF(p)(v)$ depends only on $c'(0)$, which is v, and not on the curve chosen. Also, the linearity is evident from the formula. This completes the proof of the theorem.

Items 79-83 were done on 8-8-2006.

84. We looked at some concrete examples of surfaces and maps between them. We also computed the derivatives of the maps:

(a) Let $S_1 = \{z = 0\}$ and $S_2 := \{ax + by + cz = d\}$ with $c \neq 0$. Let $F(x, y, 0) = (x, y, (d - ax - by)/c)$.

(b) Let $S_1(a, R) := \{x \in \mathbb{R}^3 : \|x - a\| = R\}$ and $S^2 \equiv S_2 := S(0, 1)$, the unit sphere. Let $F(x) := (x - a)/R$.

(c) Let $S_1 := S(0, 1)$ and S_2 be the ellipsoid $\{(x/a)^2 + (y/b)^2 + (z/c)^2 = 1\}$. Let $F(x, y, z) := (ax, by, cz)$.

(d) Let $S_1 := \{x^2 + y^2 - z^2 = 0, z > 0\}$ and S_2, the unit sphere. Let $F(x, y, z) := (x/z, y/z, 0)$. Before computing, we observed that the rank of $DF(p)$ could be at most 1.

(e) **Normal or Gauss map.** Given a surface $S \subset \mathbb{R}^3$, if (V, φ, U) is any patch, then we have a unit normal defined on U as follows: $N_p := \frac{X_u \times X_v}{\|X_u \times X_v\|}$. The map $N: U \to S^2$ given by $N(p) := N_p$ is called the normal map or the Gauss map on U.

   i. If S is the plane $\{ax + by + cz = 1\}$, the normal map $N: p \mapsto (a, b, c)$ is a constant and hence $DN(p) = 0$.

   ii. The map in Item 84b is 'the' normal map defined on all of $S(a, R)$.

   iii. If $S := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\}$ is the cylinder, then the normal map is $N: (x, y, z) \mapsto (x, y, 0)$. Again without computing we could see that the rank of DF is at most 1.

(f) Karan: To appear later! Please remind me to include it here!

(g) **Exercise.** Compute the derivatives of the maps in Item 77.

85. We recalled the notion of a length of a curve as learnt in complex analysis. We then motivated and defined the length of a continuously differentiable curve $c: [a, b] \to \mathbb{R}^n$ as $\ell(c) := \int_a^b \|c'(t)\| \, dt$. One can verify that the length of a line segment joining x and y is $\|y - x\|$ and the length of a circle $t \mapsto (R\cos t, R\sin t)$ is $2\pi R$.

86. We saw that we could extend the definition of the length of a curve c in a surface $S \subset \mathbb{R}^n$ exactly the same way.

87. What made the definition of the length of a curve in a surface possible is the availability of the dot product on the tangent spaces T_pS: The dot product on $\mathbb{R}^n$ induces a dot product on the vector subspaces T_pS. The first fundamental form on a surface is the dot products on its tangent spaces.

88. If (V, φ, U) is a chart of a surface S, then the knowledge of $E := X_u \cdot X_u$, $F := X_u \cdot X_v$ and $G := X_v \cdot X_v$ completely determines the dot product on the tangent spaces T_pS for $p \in U$. In classical notation, $I \equiv Edu^2 + 2Fdudv + Gdv^2$ is called the first fundamental form of the surface S with respect to the chart (V, φ, U). We shall denote it by I. Note that the 'coefficients' E, F, G of the first fundamental form depend on the chart chosen. See the first example in the next item.

89. We compute the first fundamental forms of the following surfaces with the given charts.

   (a) Let S be (a part of) the plane $z = 0$ with the standard parametrization $(u, v) \mapsto (u, v, 0)$ and with the polar coordinate parametrization $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$. Then the fundamental forms are $du^2 + dv^2$ and $dr^2 + r^2 d\theta^2$. Note that $F = 0$ means that the u and v-curves (respectively, r and θ curves) meet orthogonally.

(b) Let S^2 be given the standard polar parametrization: $(\cos u \cos v, \cos u \sin v, \sin u)$. Then $I = du^2 + \cos^2 u dv^2$. Here also the coordinate curves are orthogonal and we understood the meaning of the coefficient '$\cos^2 u$ by looking at the v-curves, which are the latitudes.

Items 84-89 were done on 10-8-2006.

90. We wanted to prove the Theorem in Item 78. A complete proof was given in the notes on "Four Applications ...". To get an idea of the proof, we looked at two special cases:

    (a) Let S be the plane $z = 0$ and $f\colon S \to \mathbb{R}$ be given. We can take $U = \mathbb{R}^3$ and define $g(x, y, z) = f(x, y, 0)$. Then g extends f. (On the way, we also looked at a set-theoretic problem. If $f\colon X \to Z$ is given, we have a natural map $g\colon X \times Y \to Z$ given by $g(x, y) = f(x)$, that is, $g := f \circ \pi_X$. Note that if all the sets are topological spaces and f is continuous, then g is continuous when $X \times Y$ is given the product topology!)

    (b) Let S be the unit sphere and $f\colon S \to \mathbb{R}$ be given. We take $U := \mathbb{R}^3 \setminus \{0\}$ and define $g(x) := f(x/\|x\|)$. Then g extends f and is smooth if f is smooth.

91. We understood what was happenening in the above examples. In the second, we have the spherical polar coordinates on $\mathbb{R}^3$ and S^2 is given as $r = 1$. Hence if $x \in U$ is of the form $x = (r, \xi)$, with $\xi \in S^2$, then we defined $g(x) = g(r, \xi) = f(\xi)$. Thus in each of the cases, the surface was (locally) given as the set of points $(\xi_1, \ldots, \xi_n)$ with $\xi_n = 0$, say. Then we defined the extension $g(\xi) := f(\xi_1, \ldots, \xi_{n-1}, 0)$.

92. With the idea above, all of you went through the proof of the theorem given in the notes.

93. We went back to the first fundamental form. We computed it for (i) graph surfaces and (ii) for surfaces of revolution. In the case of the surface of revolution got by revolving the curve $(f(u), 0, g(u))$ around the z-axis, we got $E = f'^2 + g'^2$ and $G = f^2$. Since E, G are norm-squareds of nonzero vectors X_u, X_v, it behoves us to stipulate that the profile curve satsifies two conditions: (i) $f(u) \neq 0$ and hence we may assume that $f(u) > 0$ and (ii) $f'^2 + g'^2 \neq 0$. Note that the second condition says that the tangent vector $(f', 0, g')$ of the profile curve is nonzero.

94. In a different direction, we arrived at the conditions for the profile curve of a surface of revolution by computuing the square of the norm of the vector product $X_u \times X_v$.

95. **Exercise.** Compute the coefficients E, F, G for the surfaces in Item 73. Do all of them, as it would improve the speed of your computational skills! *Incentive:* My intention is to ask one question of this kind in every test!

96. Let c be a curve in S. Let $s(t) := \int_a^t \|c'(\tau)\| \, d\tau$. Then $s(t)$ is the length of the curve c restricted to the interval $[a, t]$. In terms of local coordinates, we have

$$c'(t) = \frac{d}{dt}(x(u(t), v(t)), \ldots) = \left(\frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}, \ldots\right) = X_u \frac{du}{dt} + X_v \frac{dv}{dt},$$

so that

$$c'(t) \cdot c'(t) = E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2.$$

Hence $\frac{ds}{dt} = \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2}$. In classical notation this is denoted as

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2.$$

and is called the arc-length form of the first fundamental form. Refer to (i) Section 5.1 of Pressley, pages 97-100 and/or (ii) Sectin 2.5 of do Carmo, pages 92-99 for the first fundamental form and arc-length.

97. A feeble attempt was made to explain the formula for ds^2. We can think of $\{du, dv\}$ as a basis of $(T_pS)^*$ dual to the basis $\{X_u, X_v\}$ of T_pS. Then the RHS of the formula for ds^2 is the quadratic form on T_pS for $p \in U$, the parametrized neighbourhood. We shall came back to this later.

98. We ended the class with a question: What is the next concept I would like to define on a surface? One of you came with the answer: the concept of area.

Items 90-98 were done on 12-8-2006.

99. **Area element of a surface:** The most important idea in assigning area/volume etc is the geometric meaning of the determinant. We recall some of the facts which we need. For details, refer to my book on Linear Algebra.

   (a) Given $\mathbb{R}^n$ with the dot-product, and an O.N. basis $\{e_i : 1 \le i \le n\}$ of $\mathbb{R}^n$, the unit cube $[e_1, \ldots, e_n]$ is the subset $\{\sum_{i=1}^n t_i e_i : 0 \le t_i \le 1\}$. More generally, the parallelepiped $[v_1, \ldots, v_n]$ spanned by n-vectors is the set $\{\sum_{i=1}^n t_i v_i : 0 \le t_i \le 1\}$.

   (b) The *oriented volume* of the parallelepiped$[v_1, \ldots, v_n]$ is by definition

$$\text{Oriented Volume of } [v_1, \ldots, v_n] = \det(v_{ij}), \text{ where } v_i = \sum_{j=1}^m v_{ij} e_j.$$

   The volume of the parallelepiped is the absolute value of the oriented volume. Note that if the vectors are linearly dependent the parallelepiped lies on an $(n-1)$-dimensional vector subspace and hence its n-dimensional volume should be 0, and such is the case.

   (c) The volume of the parallelepiped is also computed as

$$\text{Volume of } [v_1, \ldots, v_n] = \sqrt{\det(\langle v_i, v_j\rangle)}. \tag{2}$$

   To see this, consider $A \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $Ae_i = v_i$ and compute $\det(AA^t)$.

   (d) Let $A \colon \mathbb{R}^n \to \mathbb{R}^n$ be linear. Let $\{v_i\}$ be *any* basis of $\mathbb{R}^n$. Then $\det A$ is given by

$$\det(A) = \frac{\det(\langle Av_i, v_j\rangle)}{\det(\langle v_i, v_j\rangle)}. \tag{3}$$

   For Items 99a–99d, refer to my book "Linear Algebra–A Geometric Approach", especially, Section 6.1, Section 6.2, pages 131-8, and Section 6.4.3, pages 158-9

(e) If $S \subset \mathbb{R}^n$ is a surface, then the area element dA (or dS) of on any region in a parametrized open set $U \subset S$ is defined to be Volume $[\partial_u, \partial_v]$. In the case when $n = 3$, using (2), the area element is given by $dS = \sqrt{EG - F^2}\, dudv$. That is, if $E \subset U$ is any open/closed set, then the area of E is $\int_{\varphi^{-1}E} \sqrt{EG - F^2}\, dudv$. In classical text-books, the area element is defined as $dS = \|X_u \times X_v\|\, dudv$, which is same as ours.

(f) We recalled the change of variable formula in the following form: *Let $\varphi\colon U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Let $S \subset U$ be a 'nice' set. Then the volume of $\varphi(E)$ is given as*

$$\text{Volume}\, \varphi(E) := \int_E |\det(D\varphi(x))|\, dx_1 \cdots dx_n.$$

The change of variable formula assures us that the definition of the area of $E \subset S$ is well-defined.

(g) As an example, we work out the area element for a surface of revolution got by revolving a curve $(f(u), 0, g(u))$, $a < u < b$, in the xz-plane around the z-axis with $f > 0$. We get

$$\text{Area}\,(S) = 2\pi \int_a^b f(u)\sqrt{f'^2 + g'^2}\, du.$$

If the curve is given as a graph $(u, 0, g(u))$, the formula becomes $2\pi \int_a^b u\sqrt{1 + g'^2}\, du$. This takes the familiar form in the case off revolving the curve $y = y(x)$ about the y-axis.

Special cases are: spheres, cylinders and cones. Check that we get the standard formulas for their areas.

For Items 99e–99g, refer to Section 5.4 of Pressley, pages 112-114 and pages 97-99 of do Carmo.

100. **Curvature of surfaces.** We developed intuition by looking at the familiar surfaces like planes and spheres of various radii. We decided that

(a) a plane is not curved at all and hence the curvature must the constant 0,

(b) a sphere is uniformly curved, that is, curved the 'same amount' at any point so that the curvature must be a constant,

(c) given two spheres S_r, S_R of radii $r < R$, we expect that S_R to be less curved whereas S_r to be more curved.

We started looking for a function $\kappa\colon S \to \mathbb{R}$ with these properties. We also looked at an arbitrary surface drawn on the green-board and 'compared' the curvatures at various points.

101. Now, how do I know whether or not the surface S is curved at a point $p \in S$? Many of you came with the answer that we should see how far it 'deviates' from the tangent plane at p.

Items 99-101 were done on 16-8-2006.

102. Provisionally, it is the rate of change of the tangent space, that is, the rate of change of $q \mapsto T_qS$. We found that it is easier to work with unit normal vectors in place of tangent spaces. On any patch (V, φ, U), the map $q \mapsto \frac{X_u \times X_v}{\| X_u \times X_v \|}$ is a smooth normal field on U.

103. After a little thought we arrived at the following qualitative definition of curvature. It is the rate of change of 'the' unit normal field $q \mapsto N_q$. (This map is called the normal map or the Gauss map.) We looked at some special cases (See also Item 84e):

    (a) Plane given by $ax + by + cz = d$. Here the map is a constant $p \mapsto (a, b, c)$ and hence $DN(p) = 0$.

    (b) Sphere $\|x - a\| = R$. Here the map is $x \mapsto (x - a)/R$. $DN(p)(v) = v/R$. We observed that on a sphere, the rate of change of normal field is 'a constant' and that on a sphere of larger radius, it is smaller as signified by the presence of the factor $1/R$.

    (c) On the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$, we found $N(x, y, z) = (x, y, 0)$. The tangent space is spanned by $(-y_0, x_0, 0)$ and $e_3 = (0, 0, 1)$ corresponding to the curves $c_1 := t \mapsto (\cos(t_0 + t), \sin(t_0 + t), z_0)$ and $c_2 := t \mapsto (x_0, y_0, z_0 + t)$. The rate of change of N along c_i are given by

$$\begin{aligned} \frac{d(N \circ c_1)}{dt}|_{t=0} &= (-\sin t_0, \cos t_0, 0) \\ \frac{d(N \circ c_2)}{dt}|_{t=0} &= 0. \end{aligned}$$

    In all these cases, we found that $DN(p)(v) \in T_pS$, that is, $DN(p)$ maps T_pS to itself.

    We also saw that the characteristics of a curvature function as desired in Item 100 were exhibited in these computations.

104. **Proposition.** *The derivative* $DN(p) \colon T_pS \to T_{N(p)}\mathbb{R}^3 = \mathbb{R}^3$ *takes values in* T_pS. Idea of the proof: Differentiate $\langle N(c(t), N(c(t)\rangle = 1$ where $c(0) = 0$ and $c'(0) = v$. $\square$

105. We reviewed quite a bit of linear algebra of inner product spaces: self-adjoint maps, eigenvalues, characteristic values and finally the spectral theorem for a self-adjoint map of a finite dimensional real inner product space to itself.

    (a) Definitions of eigenvalue and characteristic value of a linear map $A \colon V \to V$. A characteristic value of A is a root/zero of the characteristic equation $\det(A - \lambda I) = 0$.

    (b) Any eigenvalue of A is a characteristic value.

    (c) Converse is not true in general. For example, the rotation by $\pi/2$ represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has characteristic values $\pm\sqrt{-1}$.

    (d) If V is a complex vector space, then any characteristic value is an eigenvalue. For, if $A \colon V \to V$ is given and $\lambda \in \mathbb{C}$ is a characteristic value, then $A - \lambda I$ is a linear endomorphism of V such that $\det(A - \lambda I) = 0$. Hence $A - \lambda I$ is singular and so there exists a nonzero $v \in V$ such that $(A - \lambda I)v = 0$.

(e) If V is a complex inner product space and $A\colon V \to V$ is self-adjoint (that is $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in V$), then any characteristic value $\lambda \in \mathbb{C}$ is real. For, by the last item, λ is an eigenvalue, say, with an eigenvector v of unit norm. We have

$$\lambda = \langle Av, v\rangle = \langle v, Av\rangle = \langle v, \lambda v\rangle = \overline{\lambda}.$$

(f) **Spectral Theorem.** *Let V be a real inner product space of dimension n. Let $A\colon V \to V$ be self-adjoint. Then there exists an orthonormal basis $\{v_i : 1 \le i \le n\}$ of V such that all v_i's are eigenvectors of A.* The proof is outlined as follows:

i. By induction on the dimension.

ii. Apply the method of Lagrange multipliers to the function $f(x) := \langle Ax, x\rangle$ with the constraint $g(x) \equiv \langle x, x\rangle - 1 = 0$. The gradients are $\operatorname{grad} f(x) = 2Ax$ and $\operatorname{grad} f(x) = 2x$.

iii. The point of maximum, say, x_0 for f on the sphere $g = 0$ is an eigenvector with eigenvalue $\lambda = f(x_0)$:

$$Ax_0 = \lambda x_0 \implies f(x_0) = \langle Ax_0, x_0\rangle = \lambda.$$

iv. If $W := x_0^{\perp}$, then W is an $(n-1)$-dimensional real inner prodcut space. A maps W to itself: if $w \in W$,

$$\langle Aw, x_0\rangle = \langle w, Ax_0\rangle = \langle w, \lambda x_0\rangle = 0.$$

(g) If V as in the last item is of dimension 2, then the vectors in the eigenbasis are points at which f attains its maximum and minimum.

(h) If A is self-adjoint on a real inner product space, then the characteristic values are real. For, according to the spectral theorem, there are n eigenvalues. Since any eigenvalue is a charactertic value, the characteristic equation already has n real roots. Hence these are the only roots of the characteristic equation.

106. We defined $L_p v := -DN(p)v$ for $v \in T_pS$, called the Weingarten map at p. We proved that it is self-adjoint by showing that

$$\langle LX_u, X_v\rangle = \langle N_p, X_{uv}\rangle = \langle N_p, X_{vu}\rangle = \langle LX_v, X_u\rangle.$$

Idea of the proof: Differentiate $\langle N(c(t)), X_v\rangle = 0$ where $c(t)$ is the u-coordinate curve.

Reference for the Items 100–103 is my article "Theory of Surfaces—A Rapid Introduction" (available in MTTS notes). Item 105h has almost all the details and so you do not need any reference! Item 106 may be found in the books by do Carmo or Pressley.

Items 102-106 were done on 17-8-2006.

107. **Exercise.** Compute L_p for the following:

(a) S is the saddle surface $z = xy$ and $p = (0, 0, 0)$.

(b) S is the torus (as in Item 73h) and p is any point on the $v = 0$ curve. (Understand by means of a picture what this curve is and guess what should be L_p before you start computing!)

Both the examples were discussed in the class!

108. We may think of the Gauss map as a map from $U \subset S \to S^2$. Note that $T_pS = T_{N_p}(S^2)$ and hence L_p may be interpreted as the derivative of this map. In particular, this gives rise to a geometric interpretation of $\det L_p$: It is the 'distortion' factor by which areas in S are mapped into areas in S^2. More precisely, we shall show later that

$$\det L_p = \lim_{E \mapsto \{p\}} \frac{\text{Area}\,(N(E))}{\text{Area}\,E}.$$

109. Though we succeeded in getting a qualitative definition of curvature at p as L_p, we would like to associate to each $p \in S$ a real number, to be denoted by κ_p. Since L_p is symmetric, there are two real numbers associated with it, namely, the eigenvalues, say $\lambda_i(p)$, $i = 1, 2$. These are called the *principal curvatures* at p. Their average $(\lambda_1 + \lambda_2)/2$ is called the *mean curvature* of S at p and their product $\kappa(p) := \lambda_1\lambda_2$ is know as the *Gaussian curvature* of S at p.

    (a) The principal curvatures at any point of a plane are zero.
    (b) The Principal curvatures at any point of the sphere $\|x - a\| = R$ are $1/R, 1/R$.
    (c) The principal curvatures at any point on the cylinder $x^2 + y^2 = 1$ are $0, 1$.
    (d) The principal curvatures at $(0, 0, 0)$ of the saddle surface $z = xy$ are ± 1.

110. Differential Geometry mostly deals with the Gaussian curvature for the following reasons:

    (a) The normal map is defined locally. There are two choices involved and hence the resulting Weingarten maps are negatives of each other. While Principal and mean curvatures depend on the choice of the unit normal, the Gaussian curvature is independent of the choice of the unit normal field.
    (b) Gaussian curvature has the geometric interpretation as given in Item 108.

111. To compute $\kappa(p) := \det L_p$, we make use of (3) on page 17. If we let $\ell := \langle L_pX_u, X_u\rangle$, $m := \langle L_pX_u, X_v\rangle$ and $n := \langle L_pX_v, X_v\rangle$, then we have shown in Item 106 that $\ell = \langle N_p, X_{uu}\rangle$, $m = \langle N_p, X_{uv}\rangle$ and $n = \langle N_p, X_{vv}\rangle$. Therefore we have

$$\kappa(p) := \frac{\ell n - m^2}{EG - F^2}. \tag{4}$$

112. **Exercise.** Find the Gaussian curvature of all the parametrized surfaces seen so far. Refer to (1) Section 3.3 of do Carmo, especially Examples 1, 4, and 5, (2) Section 7.1 of Pressley and (3) Section 5.4 of Kumaresan "Diff. Geom. and Lie Groups" for such computations.

    Look at the pictures of the surfaces and *try* to guess some qualitative behaviour of the curvature before computing. For instance, I mentioned that the formula for Gaussian curvature of a surface of revolution is likely to involve only the u-coordinates. In the case of an elliptic paraboloid, the origin is the point where the Gaussian curvature is maximum whereas as $z \to \infty$, the Gaussian curvature is likely to go to zero. Keeping the saddle surface as an analogy, we made a guess on the sign of the curvature at various points on a torus.

Items 107-112 were done on 19-8-2006.

113. **Quiz 2**

(a) Let a surface $S \subset \mathbb{R}^3$ be given as a level set $f^{-1}(0)$ for some $f\colon \mathbb{R}^3 \to \mathbb{R}$. Give a unit normal field on S.

(b) Let U be a parametrized neighbourhood via the parametrization (V, φ, U). Give a unit normal field on U.

(c) Write down the Gauss/normal map on $S := \{x \in \mathbb{R}^3 : \|x - a\| = R\}$. What is its derivative?

(d) Given a curve $c(u) := (f(u), 0, g(u))$ in $\mathbb{R}^3$. Write down the parametrizations of the surface S of revolution got by revolving c about the z-axis. What are conditions imposed on f and g to a get a (smooth) surface?

(e) Given a parametrization (V, φ, U) of a surface $S \subset \mathbb{R}^3$ and a $p \in U$, identify T_pS via an object related to the derivative of φ.

(f) Let $f(x, y, z) := x^2 + y^2 + z$. Exhibit $f^{-1}(1)$ as a graph surface and hence conclude S is a surface.

(g) True or False? In a surface of revolution with the standard parametrization, the u and v coordinate curves are orthogonal to each other.

(h) Fill in the blanks. The area element $\sqrt{EG - F^2}\, du dv$ for the graph surface of $(u, v) \mapsto f(u, v)$ is ....

(i) True or False? The parallelepiped $[v_1, \ldots, v_n]$ in $\mathbb{R}^n$ has nonzero volume iff the vectors $\{v_i : 1 \le i \le n\}$ are linearly independent.

(j) Give an example of a nonconstant smooth function on S^2.

(k) Define the derivative of a smooth map $F\colon S_1 \to S_2$ between two surfaces.

(l) Give an example of a surface $S \subset \mathbb{R}^3$ in which the Gauss/normal map is 1-1.

(m) Give a curve c in the cylinder in $\mathbb{R}^3$ defined by $x^2 + y^2 = 1$ so that along c, a unit normal field is a constant.

(n) Sketch the surface $S := \{x^2 + y^2 - z^2 = 0, z > 0\}$ *in your mind*. Give a point on S^2 which is not in the image of the Gauss map. What is the rank of the Gauss/normal map?

(o) True or False? The derivative $DN(p)$ of the Gauss map on a parametrized open set U of a surface S maps T_pS to itself as well as to $T_{N(p)}S^2$.

(p) What is the relation between the Gauss map $p \mapsto N_p$ and the Weingarten map L_p?

(q) Define the principal curvatures at $p \in S$. Why do they exist? How are they related to the mean and Gaussian curvatures at p?

(r) Give an examples of surfaces for which there exists a point at which the principal curvatures are (i) the same and (ii) distinct.

(s) Give examples of 2 surfaces S_1, S_2 such that the Gaussian curvatures of S_i are the same constant but their mean curvatures are different constants.

(t) True or False? If the Weingarten map at $p \in S$ is skew-symmetric, then the Gaussian curvature of S at p is zero.

Item 113 was done on 2-9-2006. 39 students appeared for the quiz. As earlier, it took 1+1 hours. (Eighteen) Students scoring 15 and above out of 25 were given chocolates. It is heartening to note that all except two have improved their score!

114. I made you compute the Gaussian curvature of two classes of surfaces:

(a) The graph surface $(u, v) \mapsto (u, v, f(u, v))$ and found that the curvature is given by

$$K(u, v) = \frac{f_{uu}f_{vv} - (f_{uv})^2}{(1 + f_u^2 + f_v^2)^2}.$$

We applied this formula to the saddle surface $z = xy$ and to the elliptic paraboloid $z = x^2 + y^2$.

(b) Surfaces of Revolution got by revolving the curve $u \mapsto (f(u), 0, g(u))$. The curvature is given by

$$K(u, v) = \frac{(f'g'' - f''g')g'}{f(f'^2 + g'^2)^2}.$$

We applied this formula to the sphere and the torus of revolution.

In both the cases, we verified some of our hunches about the behaviour of the curvature. During the computational phase, I told you how to cross-check some of our computations. This is an essential lesson one has to learn whenever one is involved with a lot of computation.

Item 114 was the only work done on 5-9-2006.

115. **Exercise.** Show that with the standard parametrization, the tangent vectors X_u, X_v of a torus are principal directions and hence find the principal curvatures of a torus.

116. A connected locally path connected space is path connected.

117. If S is a connected surface, then any two points on it can be connected by means of a piecewise smooth path.

118. Let $S \subset \mathbb{R}^3$ be a surface. A point $p \in S$ is said to be *umblic* if the principal curvatures $\lambda_i(p)$ at p are equal. Note that this means that the Weingarten map L_p must be a scalar λI where $\lambda = \lambda_i$, $i = 1, 2$.

119. Any point on a plane or on a sphere is umblic. The origin on the paraboloid $z = x^2 + y^2$ is an umblic point. No point on the saddle surface $z = xy$ could be umblic.

120. Observation: The Gaussian curvature at an umblic point is nonnegative.

121. We proved the following global theorems in differential geometry of surfaces.

(a) *If the Weingarten map L of a connected surface $S \subset \mathbb{R}^3$ is identically zero, then S is part of a plane.*

(b) *Let S be a compact surface in $\mathbb{R}^3$. There exists a point $p \in S$ at which the Weingarten map is either negative definite or positive definite so that the Gaussian curvature at p is positive.*

(c) *Let $S \subset \mathbb{R}^3$ be a connected surface all of whose points are umblic. Then S is a part of a plane or a part of a sphere.*

We proved the first two results and started that of the third. It was a good performance by you as with the aid of a few pictures we got the idea of the proofs of the second result. A complete set of notes was given. You are required to go through the proof of the third result before we meet on Tuesday (12/9/06).

Items 115–121 were done on 7-9-2006.

122. We completed the proof of Item 121c.

123. *Any compact connected surface in* $\mathbb{R}^3$ *all of whose points are umblic is a sphere.*

124. I thought of Item 123 just this morning. One of you said that a compact connected surface in $\mathbb{R}^3$ all of whose points are umblic is part of a sphere. When asked to give an example of a proper closed subset of S^2 which is also a surface, the answer was $S^2 \cap \{z \geq 0\}$. Nobody was able to answer my question: why is this set not even a topological surface? To understand this, we raised the following question: why is the set $\{(x, y) : x^2 + y^2 \leq 1 \text{ and } x > 0\}$ not homeomorphic to an open subset of $\mathbb{R}^2$?

125. We defined the second fundamental form of a surface as $\ell(du)^2 + 2m du dv + n(dv)^2$ and explained its geometric meaning. Refer to the set of notes "A few theorems in the theory of surfaces".

126. Elliptic, hyperbolic, parabolic and planar points were defined. We also proved

   (a) At an elliptic point p, sufficiently small neighbourhood of p lies on one side of the tangent plane at p.

   (b) At a hyperbolic point p, any nighbouthood of p lies on both sides of the tangent plane.

127. Isometry of surfaces: One of you came up with the definition: A diffeomorphism $F: S_1 \to S_2$ is an isometry if it preserves the lengths of curves, that s, if c is a curve in S_1, and if $\gamma := F \circ c$ is the corresponding curve in S_2, then the lengths of c and γ are the same.

   It seems intuitively obvious that if U is a parametrized set in S_1 and $U_2 := F(U)$ is the corresponding parametrized neighbourhood of S_2, then the fundamental forms of S_1 and S_2 are the same. Equivalently, $DF(p)$ must be a linear isometry of T_pS_1 onto $T_{F(p)}S_2$ for all $p \in S_1$. We shall prove them in the next class.

   Items 122–127 were done on 12-9-2006.

128. We proved the following result.

   Theorem *Let* S_1, S_2 *be two surfaces in* $\mathbb{R}^3$. *Let* $F: S_1 \to S_2$ *be a diffeomorphism. Then the following are equivalent:*
   (1) F *preseves the lengths of curves.*
   (2) *If* (V, φ, U) *is a local chart around* $p \in S_1$, *and if* $(V, F \circ \varphi, F(U))$ *is the corresponding chart around* $F(p)$, *then the first fundamental forms of* S_1 *and* S_2 *with respect to these charts are the same. In fact, the respective coefficients are the same.*
   (3) $DF(p): F_pS_1 \to T_{F(p)}S_2$ *is a linear isometry for each* $p \in S_1$. □

129. A diffeomeorphism $F: S_1 \to S_2$ satisfying any one of the equivalent conditions is called an *isomtry* of S_1 onto S_2. A local isomtery is a smooth map F such that $DF(p)$ is an isometry for each $p \in S_1$.

130. Examples of local isometries:

   (a) Let S_1 be the $z = 0$ plane and S_2 be the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$. Then the map $F(u, v) = (\cos u, \sin u, v)$ is a local isometry. If we let $S_1 := \{(u, v) : 0 < u <$

$2\pi, v \in \mathbb{R}\}$ and $S_2 := \{(x,y,z) : x^2 + y^=1\} \setminus \{(1,0,z)\}$, then the restriction of F to S_1 is an isometry.

(b) Let $S_1 := \mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}$ and S_2 be the cone $z^2 = x^2 + y^2$. Then the map $F(r\cos t, r\sin t) = (\frac{r}{\sqrt{2}}\cos\sqrt{2}t, \frac{r}{\sqrt{2}}\sin\sqrt{2}t, \frac{r}{\sqrt{2}})$ is a local isometry.

131. Gauss Theorema Egregium: *If $F\colon S_1 \to S_2$ is a local isometry, then the Gaussian curvature of S_1 at p and that of S_2 at $F(p)$ are the same.*

This theorem is stated without proof.

Items 128–131 were done on 13-9-2006.

132. A curve $c\colon [a,b] \to S \subset \mathbb{R}^n$ is said to be parametrized by arc-length if $\|c'(t)\| = 1$ for all $t \in [a,b]$.

133. Examples:

(a) A straightline $c(t) := p + tv$, $t \in [0,b]$, can be parametrized by arc-length by setting $\gamma(t) := p + \frac{t}{\|v\|}v$.

(b) The circle $c(t) := (R\cos(t/R), R\sin(t/R))$ is parametrized by arc-length.

(c) More generally, if c has constant speed, that is, if $\|c'(t)\|$ is a constant, we can 'reparametrize' c to get an arc-length parametrization.

134. Let S be the surface of revolution of a unit speed curve $u \mapsto (f(u), 0, g(u))$. Then the curvature of S satisfies the DE: $f'' + k(u)f = 0$.

This allows one to construct surfaces with prescribed constant curvature

135. A curve c in a surface S is a *geodesic* if its acceleration $c''(t)$ is proportional to the normal at $c(t)$. This means that as observed from the surface the inhabitants of the surface may not realize any acceleration along c.

136. It is important to keep the following two points in mind.

(a) It is crucial that we deal with curves c and not with their tracks, $c(a,b)$. For instance, the curves $c(t) := (t,0,0)$ and $\gamma(t) := (t^3,0,0)$ both have the x-axis as their tracks. But c is a geodesic while γ is not.

(b) The parametrization of any geodesic is proportional to its arc-length. For, differentiation $\langle c'(t), c'(t)\rangle$ we get $\langle c'(t), c''(t)\rangle = 0$, as $c''(t)$ is proportional to the normal while $c'(t)$ is a tangent. Hence $\langle c'(t), c'(t)\rangle$ is a constant.

137. If a straight line segment c lies on a surface, then c with its 'standard' parametrization is a geodesic. In particular, the generating lines, with the standard parametrization, on the cylinder are geodesic of the cylinder.

Ex: Give two geodesics through (0,0,0) on the saddle surface $z = xy$.

138. The circle $u \mapsto (\cos u, \sin u, v_0)$ is a geodesic on the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

139. We saw that no straightline segment lies on a sphere. The great circles (on the sphere S^2) are defined as the intersection of a plane passing through the origin (that is, a two dimensional vector subspace W of $\mathbb{R}^3$) with the sphere. If we choose $v, w \in W$ as an orthonomal basis of W, then $c(t) := \cos tv + \sin tw$ is the unit speed-parametrization of the great circle $W \cap S^2$. Since $c''(t) = -c(t) = -N_{c(t)}$, c is a geodesic on S^2.

140. If $v, w \in S^2$ are linearly independent, we wrote down the parametrized form of the great circle passing through v and w.

141. The only geodesics on a plane $ax + by + cz = d$ are straight line segments.

142. Let $c := (f(u), 0, g(u))$ be an arc-length parametrized curve and let S be its surface of revolution about the z-axis. Then any of the 'profile' curves is a geodesic.

143. The only geodesics on the cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$ are of the form

$$\gamma(t) := (\cos(at + b), \sin(at + b), ct + d).$$

The special case when $a = 0$ is a generating line and the special case when $c = 0$ is a circle. The curve γ is called a helix.

Items 132–143 were done on 10-10-2006.

144. We recalled the last item.

145. We started the study of curves in space. Let $c\colon J \subset \mathbb{R} \to \mathbb{R}^3$ be a a unit speed curve, that is, $\|c'(s)\| = 1$ for all $s \in J$. Then $c''(s) \perp c'(s)$. We further *assume* that $c''(s) \neq 0$. Such curves will be called regular curves. If we let $\mathbf{t}(s) := c'(s)$ and $\mathbf{n}(s) := (1/\|c''(s)\|)c''(s)$, then $\mathbf{t}$ and $\mathbf{n}$ are called the unit tangent and normal fields along c. If we let $\mathbf{b}(s) := \mathbf{t}(s) \times \mathbf{n}(s)$, then $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is an orthonormal set of vectors. This is called the Frenet frame of the curve c.

146. $\kappa(s) := \|c''(s)\|$ is called the curvature of the **unit speed curve** c at s.

147. The study of space curves revolves around the rate of change of the Frenet frame of the curve. We arrived at the following set of equations, called the Frenet (or sometimes Frenet-Serre) formulas:

$$\begin{array}{rcccc} \mathbf{t}'(s) & = & & \kappa(s)\mathbf{n}(s) & \\ \mathbf{n}'(s) & = & -\kappa(s)\mathbf{t}(s) & & +\tau(s)\mathbf{b}(s) \\ \mathbf{b}'(s) & = & & -\tau(s)\mathbf{n}(s) & \end{array} \tag{5}$$

τ is called the torsion of c at s.

148. We write (5) in matrix form

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{pmatrix}.$$

Note the skew-symmetry of the first matrix on the right. What Frenet formula tells us is the rate of change of the Frenet frame.

149. We made an attempt to understand the skew-symmetry of the matrix above. If $A\colon s \mapsto A(s)$ is a smooth curve in $O(n)$, the set of orthogonal matrices with $c(0) = I$, then $A'(0)$ is a skew-symmetric matrix. In particular, $T_I O(n)$, the tangent space at I to the set of orthogonal matrices, is a subset of the set of skew-symmetric matrices. Later, we shall se that they are equal!

150. We saw that the unit speed parametrization of the helix is given as follows:

$$s \mapsto \left(\cos\left(\frac{a}{\sqrt{a^2+c^2}}t+b\right), \sin\left(\frac{a}{\sqrt{a^2+c^2}}t+b\right), \frac{c}{\sqrt{a^2+c^2}}t+d\right).$$

Hence its curvature is $\frac{a^2}{a^2+c^2}$. Note that, as a cross-check, that the curvature of the special helices (circles and straightlines) are what we would expect them to be. The torsion of the helix is ....

Items 144–150 were done on 12-10-2006.

151. We did the following problems.

(a) If $c : (a,b) \to \mathbb{R}^3$ is a regular curve such that $\kappa = 0$, then c is a straight line.

(b) A regular curve with $\kappa(s) \neq 0$ for any s lies in a plane iff $\tau(s) = 0$ for all s.

(c) If all the tangent lines to a curve pass through a fixed point a, then the curve is a straight line.

(d) Let c be a unit speed curve lying on a sphere of radius R. Show that the curvature of c at any point is at least $1/R$.

152. Let $c\colon [a,b] \to \mathbb{R}^n$ be a curve. $\gamma\colon [\alpha,\beta] \to \mathbb{R}^n$ is said to be a *reparametrization* of c if there exists a smooth bijection $h\colon [\alpha,\beta] \to [a,b]$ with $h' > 0$ such that $\gamma = c \circ h$. The map h is called the reparametrizing map.

On the way to this definition, we also discussed the homeomorphism of intervals.

Note that the lengths of c and γ will be the same. That is, the length of a curve does not change after a reparametrization. (The length of a railway track remians the same whether the trains travels fast or slow along the track!)

153. **Theorem.** *A curve c admits a unit speed reparametrization if and only if c has nowhere vanishing speed, that is, $c'(t) \neq 0$ for any t.* □

154. We have the following formulas for the curvature and the torsion of a curve whose parametrization is not of unit speed. *Let $c\colon (a,b) \to \mathbb{R}^3$ be a regular C^3-curve, not necessarily parametrized by the arc-length. Then we have*

$$\kappa(t) = \frac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3} \tag{6}$$

$$\tau(t) = \frac{\det(c'(t), c''(t), c'''(t))}{\|c'(t) \times c''(t)\|^2}. \tag{7}$$

These formuals were not derived in the class. But the computation is given in the notes which also gives the complete details of the theory of space curves.

Items 151–154 were done on 14-10-2006.

155. The second part of the course deals with areas/volumes and orientation and culminates in Stokes theorem. The crucial ingredient from Linear Algebra for this part is the concept of determinants.

(a) The notion of oriented volume of the parallelepiped defined by $v_1, \ldots, v_n \in \mathbb{R}^n$ was recalled.

(b) Given a real vector space V with an ordered basis $\mathbf{v} = \{v_1, \dots, v_n\}$, we introduced the notion of when another ordered basis $\mathbf{w} := \{w_1, \dots, w_n\}$ has the same as $\mathbf{v}$ or the opposite orientation.

156. Recalled the defintion of a k-dimensional manifold in $\mathbb{R}^n$.

157. Examples were: (i) the graph of a smooth function f from an open set $V \subset \mathbb{R}^k$ to $\mathbb{R}^l$. (ii) The level set $S = f^{-1}(0)$ of a smooth function $f\colon \mathbb{R}^{m+k} \to \mathbb{R}^m$ where $Df(p)$ has rank m for each $p \in S$. The second example took quite some effort for us to arrive at.

158. We discussed the definition of a smooth function $f\colon S \subset \mathbb{R}^n \to \mathbb{R}$ on a k-manifold. To make our concept well-defined, we saw the need to prove an analogue of the theorem in Item 75.

159. Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ be smooth. We defined $df_p\colon T_pU \to \mathbb{R}$ as $df_p(v) := Df(p)(v) \equiv D_vf(p) =\equiv v \cdot \operatorname{grad} f(p)$. Let $x_i\colon U \to \mathbb{R}$ be defined by $x_i(q)$ as the i-th coordinate of q. We then showed that $dx_{i,p}$ form a basis of T_p^*U dual to the 'standard' basis $\{e_1, \dots, e_n\}$ of T_pU.

160. We proved that $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$.

161. We defined a (smooth) differential 1-form on U to be an expression of the form $\omega = \sum_{i=1}^n f_i dx_i$, where $f_i \in C^\infty(U)$.

162. Is any 1-form ω on U is of the form df? No, there is a necessary condition on the coefficients f_i in ω: $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ for $1 \le i, j \le n$.

We have seen this condition earlier (in a previous janma!): (i) while discussing exact differential equations, (ii) while learning Green's theorem or conservative vector fields.

163. Let $\varphi\colon U \subset \mathbb{R}^m \to V \subset \mathbb{R}^n$ be smooth. Let ω be a 1-form on V. We discussed how to 'pull-back' the 1-form ω to a 1-form on U.

$$f^*(\omega)(v) = \omega_{\varphi(p)}(D\varphi(p)(v)), \text{ where } v \in T_pU.$$

Items 155–163 were done on 30-10-2006. You were asked to bring the notes on 'Tangent Spaces' and 'Four Theorems' on 31-10-2006.

164. We recalled the defintions of df, dx_i smooth 1-forms and the pull-back $\varphi^*(\omega)$ of a 1-form under the map $\varphi\colon U \to V$, where ω is a 1-form on V.

165. Let X, Y, Z be sets and $\varphi\colon X \to Y$ be a map. Let $\mathcal{F}(X, Y)$ denote the set of maps from X to Y. Then we have a map $\varphi^*\colon F(Y, Z) \to \mathcal{F}(X, Z)$ given by $\varphi^*(f) := f \circ \varphi$ where $f\colon Y \to Z$ is a map. The map φ^* is called a pull-back.

166. Given a smooth map $\varphi = (\varphi_1, \dots, \varphi_n)\colon U \subset \mathbb{R}^m \to V \subset \mathbb{R}^n$ and a 1-form $\omega = \sum_{j=1}^n g_j dy_j$ on V, we arrived at the following:

$$\varphi^*(dy_j) = d(y_j \circ \varphi) = d\varphi_j, \tag{8}$$

$$\varphi(\omega) = \sum_{j=1}^n (y_j \circ \varphi) d(y_j \circ \varphi) = \sum_{j=1}^n (y_j \circ \varphi) d\varphi_j. \tag{9}$$

In particular, we find that

$$\varphi^*(\omega) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} (g_j \circ \varphi) \frac{\partial \varphi_j}{\partial x_i} \right) dx_i,$$

so that $\varphi^*(\omega)$ is a smooth 1-form on U.

The formulas (8) and (9) will be repeatedly used in this part of the course.

167. Concrete examples of pull-back of 1-forms:

   (a) Let $\varphi \colon \mathbb{R} \to \mathbb{R}$ be smooth with coordinates x and y on the domain and codomain respectively. Let $\omega = dy$ be a 1-form on the codomain. Then $\varphi^*(\omega) = \varphi'(x)dx$.

   (b) Let $\gamma \colon \mathbb{R} \to \mathbb{R}^2$ be defined by $\gamma(t) = (\cos t, \sin t)$ and $\omega = (-y\, dx + x\, dy)/(x^2 + y^2)$. Then $\gamma^*(\omega) = dt$.

   If we consider $U = \{(x, y) \in \mathbb{R}^2 : x \neq 0\}$ and let $\theta(x, y) := \tan^{-1}(y/x)$, then $\omega = d\theta$ on U. Thus intuitively, ω is the differential of 'the argument function'.

168. Let $\gamma \colon [a, b] \to U \subset \mathbb{R}^n$ be smooth and ω be a 1-form on U. We then define $\int_\gamma \omega$ as follows. Let $\gamma^*(\omega) = f(t)\, dt$. Then $\int_\gamma \omega := \int_a^b f(t)\, dt$, the latter being the usual Riemann integral.

   In concrete terms, if we let $\omega = \sum_{i=1}^n f_i dx_i$, then $\int_\gamma \omega = \int_a^b (f_1, \dots, f_n) \cdot \gamma'(t)\, dt$. You might have seen this is the formula in your B.Sc. The integral is the work done by the force field $F = (f_1, \dots, f_n)$ by moving a particle along γ.

169. As an explicit example, we looked at the $\omega = d\theta$ in Item 167b where we restrict the domain of γ to $[0, 2\pi]$. Then we find $\int_\gamma \omega = 2\pi$. To understand the significance of this result, see Item 171.

170. A trivial lemma: *Let ω be a 1-form on an open set $U \subset \mathbb{R}^n$. Assume that $\omega = df$ for some $f \in C^\infty(U)$. Let $\gamma \colon [a, b] \to U$ be any closed path. Then $\int_\gamma \omega = 0$.*

   For, $\int_\gamma \omega = \int_a^b \frac{d(f \circ \gamma)}{dt}\, dt = f(\gamma(b)) - f(\gamma(a))$ by the fundamental theorem of calculus. $\square$

171. In view of the last item it follows that the 1-form ω on the punctured plane cannot be the differential of any smooth function f. (This means that there exists no smooth choice of argument on the punctured plane.)

   Items 164–171 were done on 31-10-2006.

172. **Exercise.**

   (a) Let $\Omega^1(U)$ denote the set of all smooth 1-forms on an open set $U \subset \mathbb{R}^n$. Show that $\Omega^1(U)$ is a real vector space, in fact, it is a module over $C^\infty(U)$.

   (b) Let $f, g \in C^\infty(U)$. Show that $d(fg) = fdg + gdf$.

   (c) Let $\varphi \colon \mathbb{R}_+ \times \mathbb{R} \subset \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\varphi(r, \theta) = (r\cos\theta, r\sin\theta)$. Let $\omega = xdx + ydy$ be a 1-form on the codomain. Show that $\varphi^*(\omega) = rdr$.

We decided not to pursue the differential forms for a while and to come back to it later after we learned a few things about k-dimensional manifolds in $\mathbb{R}^n$.

173. We looked at the analogues of the following results. We found that the proofs given earlier in the case of surfaces (=2-dimensional manifolds) went through with minimal change of notation.

(a) Let $S \subset \mathbb{R}^n$ be a k-dimensional manifold. Let (V, φ, U) be a patch in S. Let $W \subset \mathbb{R}^m$ be open and $F \colon W \to \mathbb{R}^n$ be smooth such that $F(W) \subset U$. Then the map $\varphi \circ F \colon W \to V$ is smooth.

(b) Let $S \subset \mathbb{R}^n$ be a k-dimensional manifold. Let (V_i, φ_i, U_i), $i = 1, 2$ be two patches in S with $U_1 \cap U_2 \neq \emptyset$. Then the map $\varphi_2^{-1} \circ \varphi_1 \colon \varphi_1(U_1 \cap U_2) \to \mathbb{R}^k$ is smooth. (This is paraphrased as: the transition functions (that is, $\varphi_2^{-1} \circ \varphi_1$) are smooth.

(c) Let $S \subset \mathbb{R}^n$ be a k-dimensional manifold. Then S is locally a graph of a smooth function f defined on an open set $V \subset \mathbb{R}^k$. That is, there exists an open neighbourhood U of p in $\mathbb{R}^n$, an open set $V \subset \mathbb{R}^k$ and a smooth function $h \colon V \to \mathbb{R}^{n-k}$ such that $U \cap S = \{(x, h(x)) : x \in V\}$.

(d) Let $f \colon U \subset \mathbb{R}^{k+m} \to \mathbb{R}^m$ be a smooth function with 0 as a value. Let $S := f^{-1}(0)$ and assume that for each $p \in S$, the derivative $Df(p) \colon \mathbb{R}^{k+m} \to \mathbb{R}^m$ has maximal rank, namely m. Then S is a k-dimensional manifold. (This is an easy consequence of the implicit function theorem, as seen earlier.)

(e) Let $S \subset \mathbb{R}^n$ be a k-dimensional manifold. Let $f \colon S \to \mathbb{R}$ be smooth. Given $p \in S$, there exists an open set $U \in \mathbb{R}^n$ with $p \in U$ and a smooth function $g \colon U \to \mathbb{R}$ such that $g = f$ on $U \cap S$.

Items 172–173 were done on 1-11-2006.

174. Using the fact that any k-manifold is locally a graph, we gave an alternate proof of the following fact: Let a smooth function $f \colon S \subset \mathbb{R}^n \to \mathbb{R}$ on a k-manifold be given and $p \in S$. Then there exists an open neighbourhood of p in $\mathbb{R}^n$ and a smooth function $g \colon U \to \mathbb{R}$ such that $g = f$ on $U \cap S$.

175. We discussed the tangent spaces to a k-dimensional manifold $S \subset \mathbb{R}^n$. We established the following results which are analogous to the earlier ones about surfaces.

(a) Let S be a k-manifold which is the graph of a function $h \colon V \to \mathbb{R}^{n-k}$. Then we have the u_i-coordinate curves $c_i \colon (a + te_i, h(a + te_i)) \in S$. The tangent vector $c_i'(0)$ is given by

$$c_i'(0) = \partial_i \equiv \frac{\partial}{\partial x_i} = \left(0, \ldots, 0, 1, 0, \ldots, 0, \frac{\partial h}{\partial u_i}\right), \text{ 1 at the } i\text{-place.}$$

If c is any curve in S and if we write $c(t) = (x(t), h(x(t))$, then $c'(0) = \sum_i x_i'(0)\partial_i$. That is any tangent vector is a linear combination of the 'basic' tangent vectors ∂_i. Any such linear combination is a tangent vector at $p = (a, h(a)) \in S$.

(b) Let a k-manifold $S \subset \mathbb{R}^n$ arise as a level set as in Item 173d. Then $T_pS = \ker Df(p)$.

176. Recall that when we discussed the Frenet formulas, we showed that $T_I(O(n))$, (the tangent space at the identity of the set of orthogonal matrices) is a subset of the vector space of skew-symmetric matrices. Applying Item 175b, we now showed that $T_I(O(n))$ is the space of skew-symmetric matrices. Hint: Consider the function $F \colon M(n, \mathbb{R}) \to \mathrm{Symm}_n$ defined by $F(X) = XX^t$. Then $DF(I)(H) = H + H^t$.

177. Let $S_1 \subset \mathbb{R}^m$ and $S_2 \subset \mathbb{R}^n$ be a k-manifold and an l-manifold respectively. A continuous map $F\colon S_1 \to S_2$ is said to be smooth if for each pair of patches, the maps $\varphi_2^{-1} \circ F \circ \varphi_1\colon V_1 \to V_2$ is smooth.

178. For a smooth map as in the last item we have the derivative map $DF(p)\colon T_pS_1 \to T_{F(p)}S_2$ as in the case of surfaces.

179. We wanted to define differential 1-forms on a k-manifold $S \subset \mathbb{R}^n$. Recall that a 1-form on an open set $U \subset \mathbb{R}^m$ was defined as a 'smooth' assignment $p \mapsto \omega_p \in T_p^*U$. Here 'smoothness' meant that when w_p was expressed as a linear combination of the 'natural' basis $dx_i \in T_p^*U$, the coefficients as functions of p were required to be smooth.

180. Given $f \in C^\infty(S)$, we define $df_p \in T_p^*(S)$ as $df_p(v) = Df(p)(v)$.

181. Given a patch (V, φ, U) of S, we let $x_i\colon U \to \mathbb{R}$ be defined by $x_i(q) := \pi_i \circ \varphi^{-1}(q)$. Then x_i are smooth functions on U.

Items 174–181 were done on 2-11-2006.

182. We claim that dx_i form a basis of T_p^*S dual to the basis $\{\partial_i\}$.

Recall that if (V, φ, U) is a patch/chart in S, then $T_pS = D\varphi(q)(\mathbb{R}^k)$ where $\varphi(q) = p$, as seen in the case of a surface. Also, if we let $c_j(t) := \varphi(q + te_j)$, then c_j is the j-th coordinate curve through p. Its tangent vector at p is denoted by ∂_j or by $\frac{\partial}{\partial x_j}$. The set $\{\partial_j : 1 \leq j \leq n\}$ is a basis of T_pS. Note that $x_i \circ c_j(t) = q_i + t$ or q_i according as $j = i$ or $j \neq i$. The claim is obvious now:

$$dx_i(\partial_j) = \frac{d}{dt}(x_i \circ c_j)\,|_{t=0} = \delta_{ij}.$$

183. A smooth 1-form (or differential 1-form) on S is an assignment $p \mapsto \omega_p \in T_p^*S$ with the property that if ω is expressed as $\omega = \sum_{i=1}^n f_i dx_i$ in any local chart (V, φ, U), then f_i are smooth functions on U.

As an example, if $f\colon S \to \mathbb{R}$ is smooth then df is a 1-form as $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$ in a local patch.

184. Let V be an n-dimensional real vector space with a basis $\{v_1, \ldots, v_n\}$. Let f_i be the dual basis of V^*. Let $f_i \wedge f_j\colon V \times V \to \mathbb{R}$ be defined by

$$f_i \wedge f_j(v, w) := \det \begin{pmatrix} f_i(v) & f_i(w) \\ f_j(v) & f_j(w) \end{pmatrix}.$$

Then $f_i \wedge f_j$ is an alternating bilinear form (or a 2-form, in short) on V. The set $\{f_i \wedge f_j : 1 \leq i < j \leq n\}$ forms a basis for the vector space of all 2-forms on V.

The geometric meaning of this is best understood with an easy example. Let $V = \mathbb{R}^3$ with the standard basis. Then $f_1 \wedge f_2(v, w) = 0$ where $v = (1, 1, 5)$ and $w = (2, 2, 5)$. But, $f_1 \wedge f_3(v, w) \neq 0$. Thus $f_i \wedge f_j$ gives us the 'oriented area' of the parallelogram spanned by the projections of v and w on the plane spanned by e_i, e_j.

185. Keep the notation of the last item. If $I := \{1 \le i_1 < i_2 < \cdots < i_k \le n\}$, we let $f_I = f_{i_1} \wedge \cdots \wedge f_{i_k} \colon \underbrace{V \times \cdots \times V}_{(k-\text{times})} \to \mathbb{R}$ be the k-form defined by

$$f_I(v_1, \ldots, v_k) := \det(f_{i_j}(v_r))_{1 \le j, r \le k}.$$

The vector space $\Lambda^k(V)$ of all k-forms on V has the set $\{f_I : I \text{ as above}\}$ as a basis.

186. Let $S \subset \mathbb{R}^n$ be a k-manifold. Let (V, φ, U) be a local chart in S. Using the standard notation, we let dx_i be the dual basis of T_pS. We define $dx_i \wedge dx_j$ a 2-form on T_pM as in Item 184.

187. More generally, we defined the k-form dx_I on T_pS. A smooth k-form on S is defined to be an assignment $p \mapsto \omega_p \in \Lambda^k(T_pS)$ with the following property: given any local chart (V, φ, U) in S, if we express ω as a linear combination of dx_I's, say, $\omega = \sum_{I;|I|=k} f_I dx_I$, then $f_I \in C^\infty(U)$.

Items 182–187 were done on 3-11-2006.

188. We collect the basic facts about the algebra of differential forms below. For, $r \geq 1$, we let $\mathcal{A}^r(S)$ denote the set of all smooth r-forms on a k-manifold $S \subset \mathbb{R}^n$. We let $\mathcal{A}^0(S) = C^\infty(S)$.

189. If $\alpha, \beta \in \mathcal{A}^r(S)$, then $\alpha + \beta \in A^r(S)$ is defined in an obvious way. Note that $\mathcal{A}^r(S)$ is a real vector space. In fact, it is a $C^\infty(S)$-module: given $f \in C^\infty(S)$ and $\alpha \in A^r(S)$, we define $f \cdot \alpha \in \mathcal{A}^r(S)$ as follows:

$$(f \cdot \alpha)_p(v_1, \ldots, v_r) := f(p)\alpha_p(v_1, \ldots, v_r), v_i \in T_pS.$$

In terms of local coordinates, if $\alpha = \sum_I \alpha_I dx_I$ and $\beta = \sum_I \beta_I dx_I$, then

$$\alpha + \beta = \sum_I (\alpha_I + \beta_I) dx_I \text{ and } f \cdot \alpha = \sum_I (f\alpha_I) dx_I.$$

190. Given a k-form $\alpha = \sum_I f_I dx_I$ and an r-form $\beta := \sum_J g_J dx_J$, we define the $k+r$-form by setting

$$\alpha \wedge \beta = \sum_{I,J} f_I g_J dx_I \wedge dx_J.$$

Note that $dx_I \wedge dx_J = 0$ iff $I \cap J \neq \emptyset$. The $k+r$-form $\alpha \wedge \beta$ is called the exterior product of α and β. The exterior product has the following properties: Let α, β and ω be a k, an r and an s-form on S. Then

(a) $(\alpha \wedge \beta) \wedge \omega = \alpha \wedge (\beta \wedge \omega)$.

(b) $\alpha \wedge \beta = (-1)^{kr} \beta \wedge \alpha$.

(c) If $k = r$, then $\omega(\alpha + \beta) = \omega \wedge \alpha + \omega \wedge \beta$.

191. Let α be a 1-form on S. Then $\alpha \wedge \alpha = 0$. For a form α of general degree, it need not happen that $\alpha \wedge \alpha = 0$. For example, consider $\alpha = dx \wedge dy + dz \wedge dw$ on $\mathbb{R}^4$.

192. We let $\mathcal{A}(S) := \oplus_{r=0}^{k=\dim S} \mathcal{A}^r(S)$. If $\alpha = \sum_r \alpha_r$ and $\beta = \sum_{r=0}^k \beta_r$, then $\alpha + \beta = \sum_r (\alpha_r + \beta_r)$ and $\alpha \wedge \beta = \sum_{r,s} \alpha_r \wedge \beta_s$.

193. Let $S_i \subset \mathbb{R}^{n_i}$ be manifolds of dimension k_i. Given a smooth map $F\colon S_1 \to S_2$, we can define the pull-back F^* which takes a k-form on S_2 to a k-form on S_1 as follows:

$$(F^*\omega)(v_1, \ldots, v_k) := \omega(DF(p)(v_1), \ldots, DF(p)(v_k)).$$

We let $F^*(f) := f \circ F$ if $f \in \mathcal{A}^0(S_2)$.

194. Let y_i be local coordinates on S_2. Recal that $y_i := \pi_i \circ \varphi_2^{-1}$ on a local chart (V_2, φ_2, U_2) of S_2. We then write $F(q) = (f_1(q), \ldots, f_r(q))$ where $f_i(q) := y_i \circ F(q)$. Then $F^*(dy_i) = df_i$.

195. If $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ is given by $F(u,v) = (f_1(u,v), f_2(u,v))$, then $F^*(dx \wedge dy) = df_1 \wedge df_2 = J_F du \wedge dv$, where J_F is the determinant of the jacobian of F. More generally, if $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is smooth, then

$$F^*(dy_1 \wedge \cdots \wedge dy_n) = df_1 \wedge \cdots \wedge df_n = J_F dx_1 \wedge \cdots \wedge dx_n.$$

Items 188–195 were done on 7-11-2006.

196. Let the notation be as in the last item. The pull-back of forms has the following properties:

(a) $F^*(\alpha+\beta) = F^*(\alpha) + F^*(\beta)$ where $\alpha, \beta \in \mathcal{A}^r(S_2)$.

(b) $F^*(f\alpha) = F^*(f)F^*(\alpha)$ where $f \in \mathcal{A}^0(S_2)$.

(c) If $\alpha_1, \ldots, \alpha_k$ are 1-forms on S_2, then $F^*(\alpha_1 \wedge \cdots \wedge \alpha_k) = F^*(\alpha_1) \wedge \cdots \wedge F^*(\alpha_k)$.

197. Let $\alpha := dy_{i_1} \wedge \cdots \wedge dy_{i_k}$. As a consequence of the last item and (8) in Item 166, we find that

$$F^*(\alpha) = df_{i_1} \wedge \cdots \wedge df_{i_k}.$$

198. With the notation of Item 193, the following are true:

(a) $F^*(\alpha \wedge \beta) = F^*(\alpha) \wedge F^*(\beta)$ for any k-form α and an r-form β on S_2.

(b) * If S_3 is a manifold in $\mathbb{R}^n$ and if $G\colon S_2 \to S_3$ is smooth, then $(G \circ F)^*(\eta) = F^*(G^*(\eta))$ for a k-form η on S_3. (Exercise!)

199. **Exercise:** (a) Let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(u,v) = (x,y) \equiv (e^u \cos v, e^u \sin v)$. Compute $F^*(dx \wedge dy)$. (b) Let $F\colon (0,\infty) \times (0,\pi) \times (-\pi,\pi) \to \mathbb{R}^3$ be defined by $F(r,u,v) = (x,y,z) \equiv (r\cos u \cos v, r \cos u \sin v, r \sin u)$. Compute $F^*(dx \wedge dy \wedge dz)$.

200. We now define the exterior derivative $d\colon \mathcal{A}^r(S) \to A^{r+1}(S)$ which generalizes the map form $\mathcal{A}^0(S)$ to $A^1(S)$ given by $f \mapsto df$. Let $\alpha = \sum_I f_I dx_I$ be given in local coordinates. We define

$$d\alpha = \sum_I df_I \wedge dx_I.$$

201. To understand the exterior derivative, let us look at some special cases.

(a) If $\alpha = xdx$ on $\mathbb{R}^2$, then $d\alpha = 0$.

(b) If $\alpha = ydx$ on $\mathbb{R}^2$, then $d\alpha = -dx \wedge dy$.

(c) Let $\alpha := pdx + qdy$ be a 1-form on $\mathbb{R}^2$. Then

$$d\alpha = \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\right) dx \wedge dy.$$

The reader should compare this with the statement of Green's theorm.

(d) If $f\colon \mathbb{R}^3 \to \mathbb{R}$ is smooth, then $df = \sum_{i=1}^3 \frac{\partial f}{\partial x_i} dx_i$. (This is something like $f \mapsto \operatorname{grad} f$.)

(e) Let $\omega = f_1 dx_1 + f_2 dx_2 + f_3 dx_3$ be a 1-form. We may think of the 1-form as the vector field $F = (f_1, f_2, f_3)$.

$$\begin{aligned} d\omega &= \left(\tfrac{\partial f_2}{\partial x_1} - \tfrac{\partial f_1}{\partial x_2}\right) dx_1 \wedge dx_2 + \left(\tfrac{\partial f_3}{\partial x_2} - \tfrac{\partial f_2}{\partial x_3}\right) dx_2 \wedge dx_3 \\ &\qquad + \left(\tfrac{\partial f_1}{\partial x_3} - \tfrac{\partial f_3}{\partial x_1}\right) dx_3 \wedge dx_1. \end{aligned}$$

Thus we see that d on 1-forms can be identified with the *curl* ($\bigtriangledown$) of the vector field F.

(f) Let $\varphi = g_1\, dx_2 \wedge dx_3 + g_2\, dx_3 \wedge dx_1 + g_3\, dx_1 \wedge dx_2$ be a 2-form on $\mathbb{R}^3$. Then we see that

$$d\varphi = \left(\sum_i \frac{\partial g_i}{\partial x_i} \right) dx_1 \wedge dx_2 \wedge dx_3.$$

Thus on 2-forms d is the *divergence* operator if we use the identification ....

Thus d 'encompasses' the standard operators of the classical vector analysis.

Items 196–201 were done on 8-11-2006.

202. **Theorem.** The exterior derivative has the following properties:
    1. $d(\alpha + \beta) = d\alpha + d\alpha$, where $\alpha, \beta \in \mathcal{A}^r(S)$.
    2. $d(\omega \wedge \alpha) = d\omega \wedge \alpha + (-1)^p \omega \wedge d\alpha$, where ω is a p-form.
    3. $d(d\alpha) = 0$ for any α.
    4. $d(F^*(\alpha)) = F^*(d\alpha)$ where α is an r-form on S_2 in the notation of Item 193. □

    Note that $d^2 = 0$ is our version of the standard facts such as curl ∘ grad $= 0$ and div ∘ curl $= 0$ from classical vector analysis.

203. Significance of Differential Forms:

Items 202–203 were done on 9-11-2006.

204. An open book quiz was conducted on 11-11-2006. I also discussed about my impressions on your performance in the first internal test.

Item 204 was done on 11-11-2006.

205. We started with the unfinished Item 203.

    (a) We interpreted the Green's theorem in the plane using differential forms. Two things to note here are (i) the definition of the integral of a form and (ii) the orientation of the boundary so as the equality (without a sign botheration) holds.

    On the way, we recalled the statement of Jordan curve theorem, its relevance to the (old-fashioned) statement of the Cauchy integral formula.

    (b) Let $U \subset \mathbb{R}^n$ be open and $\omega \in \mathcal{A}^n(U)$. If we write $\omega = f(x) dx_1 \wedge \cdots \wedge dx_n$ in terms of the natural coordinates of $\mathbb{R}^n$, then we define $\int_U \omega := \int_U f(x) dx_1 \cdots dx_n$, the right side being the standard Riemann integral (if it exists).

    A diffeomorphism $F : U \to V$ is said to be *orientation preserving* if the determinant of $DF(x)$ is positive for all $x \in U$.

    Using these concepts, we reformulated the change of variable theorem in terms of differential forms in the following way: Let $F : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$ be an orientation-preserving diffeomorphism. Let $\omega = g(y) dy_1 \wedge \cdots \wedge dy_n$ be the top degree from on V. Then

$$\int_U F^*(\omega) = \int_V \omega.$$

206. Let X be a topological space. We defined the support of a scalar function $f : X \to \mathbb{R}$ by

$$\operatorname{Supp} f := \text{Closure of } \{x \in X : f(x) \neq 0\}.$$

We looked at some simple examples. We also saw that the only analytic function on $\mathbb{C}$ which has compact support is zero. Question: What is the support of the characteristic function of $\mathbb{Q}$?

207. We defined the support of $\alpha \in \mathcal{A}^r(S)$ in an obvious(?) way:

$$\operatorname{Supp} \alpha := \text{Closure of } \{p \in S : \omega_p \neq 0\}.$$

It is easy to see that if S is a k-manifold and (V, φ, U) is a local chart in which $\alpha = f(x) dx_1 \wedge \cdots \wedge dx_k$, and if the support of $\alpha \subset U$, then $\operatorname{Supp} \alpha = \operatorname{Supp} f$.

208. If S is a k-manifold, $\alpha \in \mathcal{A}^k(S)$, and if the support of α is contained in the parametrized open set U of a local chart (V, φ, U), then we define

$$\int_S \alpha := \int_V f(x)\, dx_1 \cdots dx_k, \text{ where } \alpha = f(x) dx_1 \wedge \cdots \wedge dx_n \text{ in } U.$$

One needs to check that this is well-defined: if the support of α is compact and contained in U_i, $i = 1, 2$, parametrized open sets, then using an obvious notation, we need to claim that $\int_{V_1} \varphi_1^*(\alpha) = \int_{V_2} \varphi_2^*(\alpha)$. This is true if we further assume that the transition map $\varphi_2^{-1} \circ \varphi_1 \colon \varphi_1^{-1}(U_1 \cap U_2) \to \varphi_2^{-1}(U_1 \cap U_2)$ is orientation preserving. Assuming this, the equality of the integrals was an easy consequence of the fact that $\varphi_1^*(\alpha) = (\varphi_2^{-1} \circ \varphi_1)^*(\varphi_2^*(\alpha))$.

209. The last item led us to the following definition. A manifold S is said to be *orientable* if there exists a family of charts $\{(V_i, \varphi_i, U_i) : i \in I\}$ such that $\{U_i : i \in I\}$ is an open cover of S and the tarnsition maps (whenever they make sense) are orientation preserving. Such a family of charts is said to be *coherent*.

Items 205–209 were done on 14-11-2006.

210. We recalled the change of variable formula in terms of differential forms, integral of k-form whose support is contained in a parametrized open set U and how the notion of orientable manifolds enter the theory of integration of differential forms.

211. I explained how and why the definition $\int_b^a f(t)\, dt = -\int_a^b f(t)\, dt$ is made.

212. I briefly explained the notion of partition of unity. More on this later.

213. We wanted to define smooth functions on $\mathbb{R}^n$ with compact support. Towards this end, we showed that the function $f(t) = \begin{cases} e^{-1/t} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$ is a smooth function with $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}$. (In particular, the Taylor series of f around $t = 0$ is zero.) The main steps are as follows:

(a) Observe that $e^x > \frac{x^k}{k!}$ for $k \in \mathbb{N}$.

(b) Prove that if $x > 0$, $f(x) < k! x^k$ for $k \in \mathbb{N}$ and hence conclude that f is continuous at $x = 0$.

(c) Prove by induction that for $x > 0$, $f^{(k)}(x) = p_k(x^{-1}) f(x)$ for some polynomial of degree less than or equal to $2k$ (for $x \neq 0$). Note that

$$\begin{aligned} \left| \frac{f^{(k)}(x) - f^{(k)}(0)}{x} \right| &= \left| f(x) x^{-1} p_k(x^{-1}) \right| \\ &\leq n! x^{n-k}, \text{ for all } n > 2k. \end{aligned}$$

Conclude that $f^{(k+1)}(0)$ exists and hence f is infinitely differentiable on all of $\mathbb{R}$.

214. We recalled the Urysohn's lemma and its proof in the case of a metric space. Adapting the same trick, we constructed the function g_ε as follows. $g_\varepsilon(t) := f(t)/(f(t)+f(\varepsilon - t))$ for $t \in \mathbb{R}$. Then g_ε is differentiable, $0 \leq g_\varepsilon \leq 1$, $g_\varepsilon(t) = 0$ iff $t \leq 0$ and $g_\varepsilon(t) = 1$ iff $t \geq \varepsilon$.

215. Let f, g be as in Item 214. For $r > 0$ and $x \in \mathbb{R}^n$, define $\varphi(x) := 1 - g_\varepsilon(\|x\| - r)$. Then φ is smooth and has the following properties:
(i) $0 \leq \varphi \leq 1$, (ii) $\varphi(x) = 1$ iff $\|x\| \leq r$ and $\varphi(x) = 0$ iff $\|x\| \geq r + \varepsilon$.
Except one of you, others could not come up with the definition of φ.

Items 210–215 were done on 16-11-2006.

216. We redid the last item and found that $g(x) := f_\varepsilon(R - \|x\|)$ (where $R - r = \varepsilon$) has the properties:
(i) $0 \leq g \leq 1$,
(ii) $\varphi(x) = 1$ iff $\|x\| \leq r$ and $\varphi(x) = 0$ iff $\|x\| \geq R = r + \varepsilon$,
(iii) g is smooth on $\mathbb{R}^n$. (Why is g smooth at the origin?)

217. Given a nonempty open set $V \subset \mathbb{R}^k$, there exists a diffeomorphism F of the form $\mu_t \circ \tau_q$ such that $F(V)$ is an open set that contains $B(0,3)$. Here $\mu_t(x) = tx$ for $r \in \mathbb{R}^*$ and $\tau_q(x) := x - q$.

218. **Exercise.** Let K be a nonempty compact subset of $\mathbb{R}^n$ and let U be an open set containing K. Show that there exists a smooth function f such that $0 \leq f \leq 1$ and $f = 1$ on K and zero outside U. (Note that this is a smooth version of Urysohn's lemma with the extra hypothesis that one of the closed sets is compact.)

219. Given a k-manifold $S \subset \mathbb{R}^n$, there exists a family of local charts $\{(V_p, \varphi_p, U_p) : p \in S\}$ such that $p \in U_p$ and $B(0,3) \subset V_p$ for $p \in S$.

220. We proved the existence of a smooth partition of unity subordinate to a given open cover.
Let $S \subset \mathbb{R}^n$ be a compact k-manifold. Let $\{(V_\alpha, \varphi_\alpha, U_\alpha) : \alpha \in I\}$ be family of local charts such that $\cup_\alpha U_\alpha = S$. Then there exists a finite number f_i, $1 \leq i \leq m$, of smooth functions on S with the following properties:
(i) $0 \leq f_i(x) \leq 1$ *for all* $x \in S$.
(ii) $1 = \sum_{i=1}^m f_i(x)$ *for all* $x \in S$.
(iii) *For each* i, *there exists* α_i *such that* $\operatorname{Supp} f_i \subset U_{\alpha_i}$. □

221. On the way, we also went through a proof of the uniform continuity of a continuous function on a compact metric space to another metric space. You were asked to go through the proof and improve upon it.

Items 216–221 were done on 18-11-2006.

222. Let S be an orientable manifold. Fix an atlas $\{(V_\alpha, \varphi_\alpha, U_\alpha)\}$ of coherently oriented charts. S along with this atlas is said to be *oriented*. Let $\omega \in A^k(S)$ be of compact support. Let $\{f_1, \ldots, f_n\}$ be a finite partition of unity subordinate to the covering $\{U_\alpha\}$ of $K := \operatorname{Supp} \omega$. We define
$$\int_S \omega := \sum_{j=1}^n \int_S f_j \omega.$$

223. Let $f: \mathbb{R} \to \mathbb{R}$ be smooth with compact support, say, contained in $(-R, R)$. Then $\int_{\mathbb{R}} df = \int_{\mathbb{R}} f'(t)dt = \int_{-R}^{R} df = 0$.

224. **A simple version of Stokes theorem.** *Let S be an oriented k-manifold and let $\alpha \in A^{k-1}(S)$ be compact support. Then $\int_S d\alpha = 0$.* □

Sketch of a proof. Using a partition of unity, we write $\alpha := \sum_{j=1}^n f_j \alpha$. On a chart in which f_j has support, we have

$$f_j \alpha = \sum_{i=1}^{k} g_i dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_k.$$

Hence $d(f_j\alpha) = \sum_{i=1}^k (-1)^{i-1} \frac{\partial g_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_k$. Using Fubini, we see that

$$\int_S \frac{\partial g_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_k = \int_{x_r \neq x_i} \left(\int_{\mathbb{R}} \left(\frac{\partial g_i}{\partial x_i} \right) dx_i \right) dx_1 \cdots \hat{dx_i} \cdots dx_k = 0.$$

225. Let V be an n-dimensional real vector space. Let $\mathbf{v} := \{v_1, \ldots, v_n\}$ be an ordered basis. Let another order basis $\mathbf{w} := \{w_1, \ldots, w_n\}$ be given. If $\mathbf{v}$ and $\mathbf{w}$ are of opposite orientation, then if we define $\mathbf{w}' := \{-w_1, w_2, \ldots, w_n\}$ or $\mathbf{w}' := \{w_2, w_1, w_3, \ldots, w_n\}$, then $\mathbf{v}$ and $\mathbf{w}'$ define the same orientation on V.

226. If S is a k-manifold which admits a single parametrization, then S is orientable. In particular, any graph manifold is orientable.

227. If S admits two charts, say, (V_i, φ_i, U_i), $i = 1, 2$ such that (a) $S = U_1 \cup U_2$ and (b) $U_1 \cap U_2$ is connected, then S is orientable. In particular, S^n is orientable.

228. **Exercise.** If S is a connected manifold and if it is orientable, then there are exactly two orientations.

229. **Theorem.** *A k-manifold is orientable iff there exists a smooth k-form ω which is nowhere vanishing, that is, $\omega_p \neq 0$ for $p \in S$.* □

Items 222–229 were done on 21-11-2006.

230. Sketch of a proof of Item 229. Let ω be a nowhere vanishing k-form on S. Let $\{(V_i, \varphi_i, U_i)\}$ be an atlas of S such that U_i's are connected.. In a typical chart (V, φ, U), if we write $\omega = f(x)dx_1 \wedge \cdots \wedge dx_k$, by composing with $x_1 \mapsto -x_1$, we may assume that $f > 0$ on this chart. So we assume that the charts have this property. We claim that any two intersecting charts are coherently oriented. For, if $\omega = g(y)dy_1 \wedge \cdots \wedge dy_k$ and $\omega = f(x)dx_1 \wedge \cdots \wedge dx_k$, then

$$\begin{aligned} \omega &= g(y)dy_1 \wedge \cdots \wedge dy_k \\ &= g(y(x)) \det(\partial y_i/\partial x_j) dx_1 \wedge \cdots \wedge dx_k. \end{aligned}$$

This implies that the jacobian is positive.

Conversely, let $\{(V_\alpha, \varphi_\alpha, U_\alpha)\}$ be a oriented atlas of S. Let $\omega^\alpha := dx_1^\alpha \wedge \cdots \wedge dx_k^\alpha$. Let $\{f_\alpha\}$ be a partition of unity subordinate to the cover $\{U_\alpha\}$ of S. Define $\omega := \sum_\alpha f_\alpha \omega^\alpha$. Then ω makes sense and is nowhere vanishing. Suppose $\omega_p = 0$. Then we can assume that ω_p is of the form $f_1 \omega^{\alpha_1} + \cdots + f_n \omega^{\alpha_n}$. Let us look at the simplest case $n = 2$ and using an obvious simpler notation, we have

$$\omega = f dx_1 \wedge \cdots \wedge dx_k + g dy_1 \wedge \cdots \wedge dy_k.$$

Since p is in the intersection of the domains of x and y-coordinates, we can express ω in terms of dx_i alone. We then have on this intersection

$$\omega = (f + g(y(x)) \det(\partial y_i/\partial x_j))\, dx_1 \wedge \cdots \wedge dx_k.$$

Now, since the summands in the brackets are nonnegative, $\omega_p = 0$ iff the sum inside the bracket is zero. But all the numbers in the bracket are positive. This contradiction shows that $\omega_p \neq 0$. □

231. All the hypersurfaces that arise as level sets of functions from an open set in $\mathbb{R}^n$ to $\mathbb{R}$ are orientable. This is seen as follows. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be smooth with 0 as a regular value. Let $S := f^{-1}(0)$ and $p \in S$. On the set $U_i := \{p \in S : \frac{\partial f}{\partial x_i}(p) \neq 0\}$, we have $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ as the local coordinates. On this set, we define an $(n-1)$-form as

$$\omega := (-1)^{i-1} \left(\frac{\partial f}{\partial x_i}\right)^{-1} dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n.$$

Then ω is easily seen to be a well-defined form on S which is nowhere vanishing.

232. **Theorem.** *Let $S \subset \mathbb{R}^n$ be a hypersurface, that is, an $(n-1)$-dimensional manifold. Then S is orientable iff there exists a smooth nowhere vanishing normal field on S.* □

Item 226 is an immediate corollary of this theorem.

Sketch of a proof of the theorem. Suppose S is orientable, say, with an oriented atlas $\{(V_\alpha, \varphi_\alpha, U_\alpha)\}$. Since T_pS is n-dimensional, we have exactly two choices for a unit vector that is orthogonal to T_pS. We choose the one, say, N_p such that $\{\partial_1, \ldots, \partial_n, N_p\}$ is a positively oriented basis of $\mathbb{R}^{n+1}$. We need to show that the map $p \mapsto N_p$ is smooth. To see this, we use a local description of S around a point either as a graph manifold or as a level set. In either case, we have two smooth unit normal fields and one of which coincides with our choice.

Conversely, we may assume the existence of a smooth unit normal field $p \mapsto N_p$. Given an atlas, if required, we slightly alter the orientation of a chart (V, φ, U) so that the basis $\{\partial_1, \dots, \partial_n, N_p\}$ is positively oriented in $\mathbb{R}^{n+1}$. It is essentially an exercise (see below) in linear algebra to show that after this alteration, the resulting family of charts is coherently oriented. Hence S is orientable.

Exercise. Let $V \subset \mathbb{R}^{n+1}$ be an n-dimensional vector subspace. Let $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ be bases of V. Assume that a unit vector $u \perp V$ is such that the bases $\{v_1, \dots, v_n, u\}$ and $\{w_1, \dots, w_n, u\}$ are positively oriented in $\mathbb{R}^{n+1}$. Then the given bases of V are of the same orientation.

233. Examples of non-orientable manifolds: (i) the Mobius band (see Item 73n), (ii) the projective plane (The details are given in a separate set of notes.)

Items 230–233 were done on 23-11-2006.

234. We solved the exercise in Item 232. Then we went through the proof in Item 232.

235. Existence of a smooth nowhere vanishing normal field N on an $(n-1)$-dimensional manifold in $\mathbb{R}^n$ is a global problem. For, locally, there always exists a smooth unit normal field on S, as locally it is a level set of a function from an open set in $\mathbb{R}^n$ to $\mathbb{R}$.

236. We wrote down the transition function of the atlas of 2 charts for the unit circle.

237. With the experience of Item 236, we found the transition function of the atlas for the Mobius band. We also observed that $U_1 \cap U_2$ has two connected components on of which the jacobian of the transition function was 1 and on the other it was -1. (It is worthwhile to compare and contrast this with Item 227.) This fact alone does not prove the nonorientability of the Mobius band.

238. We proved that the Mobius band is nonorientable. The details of the last two items are given in a separate sheet. (This set contains material on even dimensional real projective spaces, which are not done and will NOT be done!)

Items 234–238 were done on 25-11-2006.

239. Let $H^n := \{x \in \mathbb{R}^n : x_n \geq 0\}$ be the upper-half space in $\mathbb{R}^n$ endowed with the subspace topology. If $U \subset H^n$ is open (in subspace topology), we say that a function $f\colon U \to \mathbb{R}$ is smooth there exists an open set V in $\mathbb{R}^n$ with $U \subset V$ and a smooth function $g\colon V \to \mathbb{R}$ such that $g = f$ on U. Note that for $p \in H^n$, we have ${}_pU = \mathbb{R}^n$. the point to note is that this is true even for $p \in H^n$ with $p_n = 0$.

240. An n-dimensional manifold is a topological space M if there exists an open cover $\{U_i\}$ of M, $\{V_i\}$ of open sets in $\mathbb{R}^n$ and a family of homeomorphisms $\varphi_i\colon V_i \to U_i$ with the following property: For all i, j with $U_i \cap U_j \neq \emptyset$, the transition maps

$$\varphi_j^{-1} \circ \varphi\colon \varphi_i^{-1}(U_i \cap U_j) \to \varphi_j^{-1}(U_i \cap U_j)$$

are diffeomorphisms.

241. Any k-manifold in $\mathbb{R}^n$ is a k-manifold.

242. An n-manifold with *boundary* is defined in a similar way except that the V_i's are assumed to be open sets in H^n.

Let M be a manifold with boundary. A point $p \in M$ is said to be a *boundary point* if there exists a parametrization (V, φ, U) such that p has 0 as its last coordinate. (Exercise: Make this precise.)

243. **Lemma.** *The definition of boundary point does not depend upon the parametrization chosen.*

Sketch of a proof. Let φ_i, φ_j are such that $\varphi_i(q_i) = p$ where $q_i = (x_1, \ldots, x_{k-1}, 0)$ and $\varphi_j(q_j) = p$ where $q_j = (y_1, \ldots, y_k)$ with $y_k > 0$. If we let $U := U_i \cap U_j$, then $\varphi_i^{-1} \circ \varphi_j \colon \varphi_j(U) \to \varphi_i(U)$ is a diffeomorphism. Since $y_k > 0$, there exists an open set $W_2 \subset \{y : y_k > 0\}$ such that $q_2 \in W_2 \subset V_2$. Now, the restriction of $\varphi_i^{-1} \circ \varphi_j$ to W_w is a diffeomorphism of W_2 onto an *open* subset of $\mathbb{R}^n$ by the inverse function theorem. But then the image $\varphi_i^{-1} \circ \varphi_j(W_2) \ni q_1$ which is a subset of V_1 contains points of the form x with $x_k < 0$, a contradiction. (Draw a picture.) □

We let ∂M to be the set of boundary points of a manifold. It is called the *boundary* of M.

244. **Theorem.** Let M be an n-manifold with boundary. The boundary ∂M is an $(n-1)$-dimensional manifold. If M is oriented, then ∂M is orientable.

Sketch of a proof. Let $\{(V_i, \varphi_i, U_i)\}$ be an oriented atlas of M. Let $V_i' := \{x \in V_i : x_n = 0\}$ and $U_i' := \{p \in U_i : x_n(p) = 0\}$. Then $\{(V_i', \varphi_i\,|_{V_i'}, U_i')\}$ is an $(n-1)$-dimensional atlas for ∂M.

We need to show that if $U_i' \cap U_j' \neq \emptyset$, the transition maps have positive jacobian. Note that if $p \in U_i' \cap U_j'$, using an obvious notation $y_n(p) = 0 = x_n(p)$. We observe that the jacobian is of the form

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}.$$

To compute $\frac{\partial y_n}{\partial x_n}$, we need to take the directional derivative in the direction of e_n **in** H^n. Since $x_n(p) > 0$ iff $y_n(p) > 0$, this means that the difference quotient is positive so that in the limit, we get a nonnegative number. Since $\det(D(\varphi_j^{-1} \circ \varphi_i)) = \frac{\partial y_n}{\partial x_n} \det((D\varphi_j'^{-1} \circ \varphi_i'))$, the result follows. □

Items 239–244 were done on 28-11-2006.

245. The boundary ∂M of an oriented manifold with boundary M is orientable. With the notation of the theorem above, if $dx_1 \wedge \cdots \wedge dx_n$ defines the orientation on M, we choose the orientation on ∂M defined by $(-1)^n dx_1 \wedge \cdots \wedge dx_{n-1}$. The resulting orientation on ∂M is called the induced orientation or as orientation *compatible* with the orientation of M. This convention is needed to make the equation in Stokes theorem to be true without any sign modification. See the last equality in the proof of Stokes theorem in the next item.

246. **Stokes Theorem.** *Let M be an n-dimensional oriented manifold with boundary ∂M. Let ∂M be given the compatible orientation. Let $\omega \in \mathcal{A}^{n-1}(M)$ be of compact support. Then we have*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Sketch of a proof. We proceed as in Item 224. Let $\{(V_i, \varphi_i, U_i)\}$ be an oriented atlas of M and choose the compatible orientation on ∂M. Let $\{f_i\}$ a partition of unity subordinate to the open cover $\{U_i\}$ of M. Write $\omega = \sum_i f_i \omega$. There are two cases. One is in which the support does not intersect the boundary. This is case is precisely what Item 224 is about.

Let us look at a typical term $\int_{\partial M} f_i \omega$. If $(x_1, \ldots, x_n)$ are the coordinates on U_i, write $f_i \omega = \sum_j (-1)^{j-1} g_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$. Then we have

$$d(f_i \omega) = \sum_j \frac{\partial f}{\partial x_j} dx_1 \wedge \cdots \wedge dx_n.$$

Hence

$$\int_M d(f_i \omega) = \sum_j \int_M \frac{\partial g_j}{\partial x_j} dx_1 \wedge \cdots \wedge dx_n.$$

Since ω has compact support (which does not intersect ∂M), we see that the support of $\frac{\partial g_j}{\partial x_j} \subset (-R, R)^{n-1} \times (\varepsilon, R)$. Hence using Fubini and the fundamental theorem of calculus on each of the terms, we see that the j-term (for $j < n$) is

$$\begin{aligned}
&\int_{x_1, \ldots, \hat{x}_j, x_n} \left(\int_{-R}^{R} \frac{\partial g_j}{\partial x_j} dx_j \right) dx_1 \cdots \hat{dx_j} \cdots dx_n \\
&= \int_{x_1, \ldots, \hat{x}_j, x_n} (g_j(x_1, \ldots, x_{j-1}, R, \ldots, x_n) - g_j(x_1, \ldots, x_{j-1}, -R, \ldots, x_n) \, dx_1 \cdots \hat{dx_j} \cdots dx_n \\
&= 0.
\end{aligned}$$

A similar proof shows that the n-th integral is also zero. Since the support of ω does not intersect ∂M, $\omega = 0$ on ∂M and hence the integral $\int_{\partial M} \omega = 0$. Thus the theorem is proves in this case.

The second case where the support meets ∂M, we have

$$\begin{aligned}
&\int_M d(f_i\omega) \\
&= \int_{H^n} \left(\frac{\partial g_1}{\partial x_1} + \cdots + \frac{\partial g_n}{\partial x_n}\right) dx_1 \cdots dx_n \\
&= \sum_j \int_{x_1,\ldots,\hat{x}_j,\ldots,x_n} \left(\int_{x_j} \frac{\partial g_j}{\partial x_j} dx_j\right) dx_1 \cdots \hat{dx_j} \cdots dx_n \\
&= \sum_{j=1}^{n-1} \int_{x_1,\ldots,\hat{x}_j,\ldots,x_n} (g_j(x_1,\ldots,x_{j-1},R,\ldots,x_n) - g_j(x_1,\ldots,x_{j-1},-R,\ldots,x_n))\, dx_1 \cdots \hat{dx_j} \cdots dx_n \\
&\qquad + \int -g_n(x_1,\ldots,x_{n-1},0) dx_1 \cdots dx_{n-1} \\
&= \int -g_n(x_1,\ldots,x_{n-1},0) dx_1 \cdots dx_{n-1} \qquad (10)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\int_{\partial M} f_i\omega &= \sum_j \int_{\partial M} (-1)^{j-1} g_j dx_1 \wedge \cdots \hat{dx_j} \wedge \cdots \wedge dx_n \\
&= \int_{\partial M} (-1)^n g_n dx_1 \wedge \cdots \wedge dx_{n-1} \\
&\qquad \text{as } dx_n = 0 \text{ on } \partial M \\
&= \int_0^R (-1)^n (-1)^{n-1} g_n dx_1 \cdots dx_{n-1}. \qquad (11)
\end{aligned}$$

The result follows from (10) and (11). □

247. We now derive some of the standard theorems of vector calculus.

(a) Green's theorem is a special cases of Stokes theorem.

(b) Let M be a bounded domain with boundary ∂M as a smooth surface. Assume that ∂M is represented as a graph surface $(u, v, g(u, v))$. If $\omega = f_1 dx_2 \wedge dx_3 + f_2 dx_3 \wedge dx_1 + f_3 dx_1 \wedge dx_2$, then

$$d\omega = \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}\right) dx_1 \wedge dx_2 \wedge dx_3.$$

In terms of the local coordinates (u, v) on M, the form ω is given by

$$\begin{aligned}
\omega &= f_1 dv \wedge dg + f_2 dg \wedge du + f_3 du \wedge dv \\
&= \left(-f_1 \frac{\partial g}{\partial u} - f_2 \frac{\partial g}{\partial v} + f_3\right) du \wedge dv.
\end{aligned}$$

In particular, $\int_{\partial M} \omega = \int_V F \cdot N dudv$, where $F := (f_1, f_2, f_3)$ and N is the normal field $(-\frac{\partial g}{\partial u}, -\frac{\partial g}{\partial v}, 1)$. Also, we have $\int_M d\omega = \int_M \operatorname{div} F dx_1 dx_2 dx_3$. Thus Gauss theorem arises as a special case of Stokes theorem.

248. **Brouwer Fixed Point Theorem.** *Let $f\colon B[0,1] \to B[0,1]$ be a smooth map. Then f has a fixed point.*

Sketch of a proof. Let ω be an $(n-1)$-form on $B[0,1]$ such that $\int_{S^{n-1}} \omega = 1$. If f has no fixed point, let $\varphi(x)$ be the point of intersection of the line segment $[f(x), x]$ with the boundary. Then φ is a smooth map from $B[0,1]$ to S^{n-1}. Note that $\varphi(x) = x$ for $x \in S^{n-1}$. Now, by Stokes theorem, we have

$$\begin{aligned} 1 = \int_{S^{n-1}} \omega &\equiv \int_{S^{n-1}} \varphi^*(\omega) \\ &= \int_{B[0,1]} d(\varphi^*(\omega)) \\ &= \int_{B[0,1]} \varphi^*(d\omega) = 0, \end{aligned}$$

since $d\omega$ is an n-form on the $(n-1)$-dimensional manifold S^{n-1} and hence is zero.

Items 245–248 were done on 29-11-2006.

We met on 30-11-2006 to resolve difficulties, if any.

Second internal test was conducted on 11-12-2006.

249. The last part of the course will deal with systems of ODE. We started with the example of finding the geodesics on the unit sphere in $\mathbb{R}^3$. This turned out to be a system of second order differentail equations $x_i''(t) = -x(t)$ with the IC: $x_i(0) = p_i$ and $x_i'(0) = v_i$, $i = 1, 2, 3$, $p, v \in S^2$ with $p \perp v$.

250. A (smooth) vector field F on an open set $U \subset \mathbb{R}^n$ is a smooth map $F\colon U \to \mathbb{R}^n$. We think of this as an assignment that associates a 'tangent vector' $F(p)$ to U at p. The geometric interpretation of a vector field is to attach to each point $p \in U$, the vector $F(p)$ as a directed line segment emanating from p. A typical physical model for this concept is the velocity vector $F(p)$ in the domain of a (time-independent) fluid flow. An *integral curve* through $x_0 \in U$ is a smooth curve $x\colon (-\varepsilon, \varepsilon) \to U$ such that $x(0) = x_0$ and $x'(t) = F(x(t))$ for $|t| < \varepsilon$. This may be thought of the trajectory of a (dust) particle left at position x_0 traverses as the fluid flows.

251. Examples of integral curves:

(a) Let $U = \mathbb{R}^n$ and $F(p) = 0$ for all p. Then the integral curve through p is given by $x(t) = p$ for all $t \in \mathbb{R}$.

(b) Let $U = \mathbb{R}^n$ and $F(p) = e_1$ for all p. Then the integral curve through p is given by $x(t) = p + te_1$ for all $t \in \mathbb{R}$. More generally, if $F(p) = v$ (where $v \in \mathbb{R}^n$ is fixed) for all p, then the integral curve through p is given by $x(t) = p + tv$ for all $t \in \mathbb{R}$.

(c) Let $U = \mathbb{R}^2$ and $F(x, y) = (x, y)$. Then the integral curve through (x_0, y_0) is given by $x(t) = e^t(x_0, y_0)$ for all $t \in \mathbb{R}$.

(d) Let $U = \mathbb{R}^2$ and $F(x, y) = (-y, x)$. Then the integral curve through (x_0, y_0) is given by

$$x(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

(e) Let $U = \mathbb{R}^2$ and $F(x,y) = (x,-y)$. Then the integral curve through (x_0,y_0) is given by $x(t) = (e^t x_0, e^{-t} y_0)$ for all $t \in \mathbb{R}$.

(f) Let $U = \{x > 0\} \subset \mathbb{R}^2$ and $F(p) = e_1$. Then the integral curve through (x_0,y_0) is given by $x(t) = p + te_1$ for all $t \in (-x_0, \infty)$.

252. We observe from all these examples that if two integral curves have a point in common, then they coincide.

253. In all the examples in Item except the last one, we saw that all the integral curves were defined on all of $\mathbb{R}$. Vector fields with this property (namely, that the domain of any integral curve can be extended to $\mathbb{R}$) are known as complete vector field.

254. Given a complete vector field $F\colon U \to \mathbb{R}^n$, we define $c_p\colon \mathbb{R} \to c_p(t) \in U$ to be the integral curve with the IC $c_p(0) = p$. Let $\varphi\colon \mathbb{R} \times U \to U$ be the map defined by $\varphi(t,p) = c_p(t)$. Let $\varphi_t\colon U \to U$ be the map $\varphi_t(x) = \varphi(t,x)$. In the examples of Item 251, we have (a) $\varphi_(t,p) = p$, (b) $\varphi(p,t) = p + te_1$ etc. The following are observed.

(a) φ_0 is the identity on U.

(b) $\varphi_s \circ \varphi_t = \varphi_{t+s} = \varphi_{s+t}$.

Items 249–254 were done 12-12-2006.

255. Let $f\colon [a,b] \to \mathbb{R}^n$ be continuous. If we write $f(t) = (f_1(t), \dots, f_n(t))$, we define

$$\int_a^b f(t)\,dt = \left(\int_a^b f_1(t)\,dt, \dots, \int_a^b f_n(t)\,dt\right).$$

The following observation is very useful and may be taken as the defining equation for $\int_a^b f(t)\,dt$:

$$\left\langle \int_a^b f(t)\,dt, v \right\rangle = \int_a^b \langle f(t), v\rangle\,dt, \text{ for all } v \in \mathbb{R}^n. \tag{12}$$

256. We have the fundamental theorem of calculus: $g(x) := \int_a^x f(t)\,dt$ is differentiable with derivative $g'(x) = f(x)$ for $x \in [a,b]$.

257. In particular, if $f\colon [a,b] \to \mathbb{R}^n$ is a continuously differentiable function, then $\int_a^b f'(t)\,dt = f(b) - f(a)$.

258. We have the following fundamental inequality:

$$\left\| \int_a^b f(t)\,dt \right\| \leq \int_a^b \|f(t)\|\,dt.$$

Hint: Use (12) where $v = \int_a^b f(s)\,ds$. Apply the standard inequality $\left|\int_a^b g(s)\,ds\right| \leq \int_a^b |g(s)|\,ds$ and the CS inequality.

259. Let $f, f_n\colon [a,b] \to \mathbb{R}^N$ be a sequence of continuous functions. Write $f_n = (f_{n1}, \dots, f_{nN})$ and $f = (f_1, \dots, f_N)$. Then $f_n \to f$ uniformly on $[a,b]$ iff $f_{nk} \to f_k$ uniformly on $[a,b]$ as $n \to \infty$.

260. We have the analogue of Weierstrass M-test. With the notation of the last item, assume that $\| f_n \| \le M_n$ for all n. Then the sequence $s_n := \sum_{j=1}^n f_j$ converges uniformly to a function $s\colon [a,b] \to \mathbb{R}^N$. We say that the series $\sum_{n=1}^\infty f_n$ is uniformly convergent on $[a,b]$ to the function s.

261. If $f_n \to f$ uniformly, then $\int_a^b f_n \to \int_a^b$.

262. Let X and Y be metric spaces. A function $f\colon X \to Y$ is said to be Lipschitz if there exists $L > 0$ such that $d(f(x_1), f(x_2)) \le Ld(x_1, x_2)$ for all $x_1, x_2 \in X$. The constant L is called a Lipschitz constant of f. Note that if $L' > L$, then L' is also a Lipschitz constant of f and that any Lipschitz function is uniformly continuous.

263. Two most important examples of Lipschitz functions are (i) a linear map $A\colon \mathbb{R}^m \to \mathbb{R}^n$ and (ii) a differentiable map $f\colon U \subset \mathbb{R}^m \to \mathbb{R}^n$ with

264. f is said to be locally Lipschitz, if for each $x \in X$, there exists $r_x > 0$ such that the restriction of f to $B(x, r_x)$ is Lipschitz.

265. Two most important examples of Lipschitz functions:

   (a) Let $A\colon \mathbb{R}^m \to X$ be a linear map from $\mathbb{R}^m$ to a normed linear space. Then A is Lipschitz, since there exists $C > 0$ such that $\| Ax \| \le C \| x \|$ for $x \in \mathbb{R}^n$. (The infimum of such C's is denoted by $\| A \|$ and called the norm of A. One easily sees that $\| Ax \| \le \| A \| \, \| x \|$ for $x \in \mathbb{R}^n$.)

   (b) Let $U \subset \mathbb{R}^m$ be an open convex set. Let $F\colon U \to \mathbb{R}^n$ be differentiable. Assume that there exists $L > 0$ such that $\| DF(z) \| \le L$ for any $z \in U$. (We say that F has bounded derivatives.) Then F is Lipschitz, in view of the mean value inequality:

$$\| F(x) - F(y) \| \le \sup\{\| F(x + t(y-x)) \| : t \in [0,1]\} \, \| x - y \| \le L \, \| x - y \| .$$

266. On the way, we went though a proof of mean value theorem for function $f\colon \mathbb{R}^m \to \mathbb{R}^n$.

267. We went through a proof of the basic existence result for the initial value problem $x'(t) = F(x(t))$ and $x_0 = a$ where F is assumed to be Lipschitz. A more precise version and its proof will be worked out in the next class.

Items 255–267 were done 26-12-2006.

268. We proved the following result known as the Fundamental Existence and Uniqueness theorem for a system of ODE's:
Theorem. *Let $U \subset \mathbb{R}^N$ be open and $F\colon U \to \mathbb{R}^N$ be locally Lipschitz. Let $a \in U$. Choose $R > 0$ such that (i) $B(a,R) \subset U$ and (ii) F is Lipschitz on $B(a,R)$ with Lipschitz constant, say L. Choose $r > 0$ so that $0 < 2r < R$ and let M be a bound for F on $B[a, 2r]$. Let $\varepsilon < \min\{r/M, 1/L\}$. Then for any $b \in B(a,r)$, there exists a unique curve $x\colon (-\varepsilon, \varepsilon) \to B(a, 2r)$ such that (i) $x'(t) = F(x(t))$ for $|t| < \varepsilon$ and (ii) $x(0) = b$.* □

   The reference for this is my book on Lie groups.

269. We proved Gronwall's inequality: *Let $f\colon (a,b) \to [0,\infty)$ be a continuous function. Let A, B be nonnegative real numbers. Assume that $f(t) \le A + B\int_a^t f(s)\, ds$. Then*

$$f(t) \le Ae^{B|t|} \text{ for } \ t \in (a,b). \tag{13}$$

□

270. Exercise. Keep the notation of Gronwall's inequality. Assume that $g\colon (a,b) \to [0,\infty)$ be continuous. Assume that $f(t) \le A + \int_a^t g(s)f(s)\,ds$. What is the analogous inequality in this case?

271. The immediate consequence of Gronwall's inequality is the following result: Let x and y be the solutions of the intial value problem $x'(t) = F(x(t))$ with $x(0) = a$ and $y'(t) = F(y(t))$ with $y(0) = b$. Then we have

$$\| x(t) - y(t) \| \le \| a - b \| \, e^{L|t|}.$$

This says that the solutions of the intial value problem

$$x'(t) = F(x(t)) \text{ and } x'(0) = a \tag{14}$$

depend continuously on the intial condition $x(0) = a$. Note also that the uniqueness result follows from (14)

Items 268–271 were done 27-12-2006.

272. We recalled that (14) led to a second proof of the uniqueness of the solutions.

273. We made a series of observations.

(a) The most important observation is that the Picard's theorem is a local existence and uniqueness theorem. Solutions may exist on an interval larger than $(-\varepsilon,\varepsilon)$ as specified in the theroem. Look at the IVP: $x'(t)x(t)$ and $x(0) = 1$ on $\mathbb{R}$.

(b) If $x\colon J_1 \to U$ and $y\colon J_2 \to U$ are solutions of the initial value problem $z'(t) = F(z(t))$ and $z(0) = x_0 \in U$, then $x = y$ on $J_1 \cap J_2$ and hence we have solution $z\colon J_1 \cup J_2 \to U$.

(c) There exists a unique open interval $J(x_0)$ on which the solution of the IVP is defined: it is maximal in the sense that if $x\colon J \to U$ is any solution, then $J \subset J(x_0)$ and of course the solutions conicde on J.

(d) If γ_p denotes the unique integral curve of F through p and if $q := \gamma_p(t_0)$, then the unique intergal curve γ_q is given by $\gamma_q(t) = \gamma_p(t+t_0)$ for all t such that $t+t_0 \in J(p)$.

(e) Let $x\colon (a,b) \to U$ be an integral curve with $b < \infty$. Let $K \subset U$ be compact such that $x(t) \in K$ for $t \in (a,b)$. Then x can be extended as an integral curve to a larger domain $(a, b+\varepsilon)$. Analogous result holds if $b > -\infty$.

(f) Let $p \in U$ and let $J(p) = (\alpha,\beta)$. If $\beta < \infty$, then given any $t \in (\alpha,\beta)$ and a compact subset $K \subset U$, there exists $t_1 \in (t,\beta)$ such that $x(t_1) \notin K$.

274. We say that a vector field $F\colon U \to \mathbb{R}^N$ is *complete* if $J(p) = \mathbb{R}$ for all $p \in U$. Note that this depends on F as well as on U. For instance, consider the constant vector fields $F_i(p) = e_i$, $i = 1,2$ on the open set $U := \{(x,y) \in \mathbb{R}^2 : x > 0\}$. Then F_2 is complete while F_1 is not.

275. Let F be a complete vector field on U. Then we have a map

$$\varphi\colon \mathbb{R} \times U \to U \text{ defined by } \varphi(t,p) := \gamma_p(t),$$

where γ_p stands for the (maximal) integral curve of F through p.

276. We let $\varphi_t(p) := \varphi(t,p)$, $t \in \mathbb{R}$. The map φ or the one-parameter family $\{\varphi_t : t \in \mathbb{R}\}$ is called the *flow* of the vector field F. It has the following properties:

(a) $\varphi_0 = I$, the identity map of U.

(b) $\varphi_s \circ \varphi_t = \varphi_{s+t}$, for all $s, t \in \mathbb{R}$.

(c) $\varphi_{-t} = \varphi_t^{-1}$ for $t \in \mathbb{R}$.

The maps φ_t are homeomorphisms of U.

Items 272–276 were done 28-12-2006.

277. Let F be a locally Lipschitz vector field on $U = \mathbb{R}^N$. Assume that the maximal integral curve $x\colon J \to \mathbb{R}^N$ has the property that for any finite open subinterval I of J, there exists a compact subset K such that $x(I) \subset K$. Then $J = \mathbb{R}$.

278. Let $F\colon \mathbb{R}^N \to \mathbb{R}^N$ be a vector field. Assume that L, M are positive constants such that $\|F(x)\| \leq L\|x\| + B$. Then F is complete. Hint: Estimate $x(t)$ as $t \in (0,T)$, say. Observe that Gronwall's inequality can be applied to conclude that $x(0,T)$ lies in a bounded set.

A typical example: Let $A\colon \mathbb{R}^N \to \mathbb{R}^N$ be linear and $b \in \mathbb{R}^N$ be fixed. Define $F(x) = Ax + v$.

279. The vector fields we considered so far are independent of the 'time variable'.. The corresponding ODE/IVP is called an autonomous system.

A continuous function $F\colon [a,b]\times U \to \mathbb{R}^N$ is said to be 'unifromly Lipschitz' in x varibale if there exists $L > 0$ such that

$$\|F(t,x) - F(t,y)\| \leq L\|x-y\| \text{ for all } t \in [a,b] \text{ and } x, y \in U.$$

F (or the corresponding ODE/IVP) is said to be nonautonomous.

Exercise: Almost whatever we said about autonomous systems can be extended to nonautonmous system.

280. We work out the Picard iteration for a special case:

$$F\begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} -y \\ x \end{pmatrix} \text{ with initial point } \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

We found that the solution (which we laready knew!) was given by $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tA}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

Items 277–280 were done 29-12-2006. **Happy New Year!**

281. We recalled the last item and introduced the exponential of a matrix. We studied some properties of this map. (On the way, we recalled some facts about the exponential function on $\mathbb{C}$, the Cauchy product of power series etc, and not to mention that the binomial expansion is valid in any ring for a pair of commuting elements!)

282. We dealt with the matrix DE: $X'(t) = A(t)X(t) + F(t)$. A complete set of notes for this and the rest of the course on ODE was given.

We compared and contrasted this proof with that of the Picard's theorem on the local existence of a solution of ODE.

Items 281–282 were done 2-1-2007.

283. We discussed the uniqueness of the matrix equation $X'(t) = A(t)X(t) + F(t)$ with $X(0) = X_0 \in M(n, \mathbb{R})$.

284. We derived the results concerning the vector valued linear ODE: $x'(t) = A(t)x(t) + f(t)$. For the last 2 items, the reference is the set of notes on Linear ODE.

Items 283–284 were done 4-1-2007.

285. The last meeting of the course was on 29 January, 2007. The answer-books (of the Internal Test 2) were shown. Some of the questions and their answers were discussed.

286. Question pattern of the final exam was explained.

287. Some general advice for success in life was given.

Items 285–287 were done 29-1-2007. I wish you all the best.