Solution of Dirichlet Problem — Perron's Method

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1 Harmonic Functions and Their Properties

Let Ω be a nonempty open subset of $\mathbb{R}^2$ or $\mathbb{C}$.

Definition 1. A real valued function $u : \Omega \to \mathbb{R}$ is said to be *harmonic* in Ω if u is in C^2 and $u_{xx} + u_{yy} = 0$.

Theorem 2. *If $f : \Omega \to \mathbb{C}$ is holomorphic and $f = u + iv$ then u and v are harmonic.*

Proof. By Cauchy Riemann equations $u_x = v_y$ and $u_y = -v_x$, so that $u_{xx} = v_{xy} = -u_{yy}$, or, $u_{xx} + u_{yy} = 0$. Apply the same argument to $-if$ to conclude that v is harmonic. □

Theorem 3. *If u is harmonic in Ω, then*
(1) u_x is the real part of an holomorphic function in Ω.
(2) If Ω is simply connected them u is the real part of an holomorphic function in Ω.

Proof. Let $f = u_x - iu_y$. Since $u \in C^2$, f is C^1.Further

$$f_y = u_{xy} - iu_{yy} = u_{yx} + iu_{xx} = if_x.$$

Thus f satisfies the Cauchy-Riemann equations and hence f is holomorphic in Ω. (1) is thus proved.

If Ω is simply connected, there is a primitive F for $f = u_x - iu_y$. Let $F = U + iV$.

$$F'(z) = U_x + iV_x = U_x - iU_y.$$

But

$$F'(z) = f(z) = u_x - iu_y.$$

Comparing the two expressions of F' we get $U_x = u_x$ and $U_y = u_y$. Hence $U(x, y) = u(x, y) + C$. Thus u is the real part of the holomorphic function $F - C$ in Ω. This proves (2). □

Corollary 4. *1.3 If u is harmonic in Ω and $h : \Omega' \to \Omega$ is holomorphic, then $u \circ h : \Omega' \to \mathbb{R}$ is harmonic.*

Proof. Fix $h(z) = z' \in \Omega$. Let $u = \Re\phi$ in a neighbourhood of z', where ϕ is holomorphic. Then $\phi \circ h$ is holomorphic in a neighbourhood of z and $u \circ h = \Re(\phi \circ h)$ in the same neighbourhood. Hence $u \circ h$ is harmonic. □

Theorem 5 (Mean Value Property for Harmonic Functions)**.** *Let u be harmonic in the disk $B(a, R)$ of radius R with centre a. Then*

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta$$

for any $0 < r < R$.

Proof. Let $u = \Re f$, f holomorphic in $B(a, R)$. then

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta,$$

by the mean value theorem for holomorphic functions. Taking real parts we get the result. □

Theorem 6 (Maximum Principle for Harmonic functions)**.** *If u is a non constant harmonic function in a connected open Ω, then u has no maximum or minimum in Ω.*

Proof. Assume that f is non constant, harmonic in Ω and that it attains its maximum M at a point a in Ω. By the mean value property

$$M = u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) d\theta,$$

for any r such that the closed disk of radius r with centre a lies entirely in Ω. Since u is continuous and $M \geq u(a+re^{i\theta})$ we have $M = u(a+re^{i\theta})$, $0 \leq \theta \leq 2\pi$. Thus u is the constant M in the disk with centre a and lying in Ω. Thus the set of points in Ω where u equals M is non-empty, and open as well as closed. Since Ω is connected we see that u is constant throughout Ω. This is a contradiction since f is non constant.

Applying the above argument to $-u$ proves the assertion concerning the minimum. □

Ex. 7. Prove Thm. 6 using the open mapping theorem for holomorphic functions and the fact that any harmonic function is locally the real part of a holomorphic function.

Remark 8. Compare and contrast Thm. 6 with the maximum (modulus) principle for a holomorphic function. In the latter case the obvious version of the minimum modulus principle has to be modified.

Corollary 9. *Let $w : \overline{\Omega} \to \mathbb{R}$ be continuous where Ω is connected and open and $\overline{\Omega}$ is compact. Assume that w is harmonic in Ω. Then w attains its maximum M and minimum m on the boundary $\partial\Omega$ of Ω. Further if u and v are two continuous real valued functions on $\overline{\Omega}$, both harmonic in Ω, and if $u = v$ on the boundary on Ω, then $u = v$ on $\overline{\Omega}$.*

Proof. If w is constant then clearly w attains its maximum and minimum on the boundary of Ω. Assume therefore that w is non constant. Since w is continuous on compact $\overline{\Omega}$, w attains its maximum M and minimum m in $\overline{\Omega}$. Since w is non constant and harmonic in Ω, the maximum and minimum of w are not attained in Ω. Hence they are attained in $\partial\Omega$. This proves the first part of the Corollary. To prove the second part put $w = u - v$, which is continuous in $\overline{\Omega}$, harmonic in Ω, and vanishes on $\partial\Omega$. Clearly by the first part of the corollary the maximum and the minimum of w are both zero in Ω, hence $u = v$ in Ω. □

Ex. 10. Show that the hypothesis of compactness of $\partial\Omega$ in the above corollary can not be dropped by exhibiting a continuous function on the closed upper half plane which is harmonic in the open upper half plane and which does not attain its maximum or the minimum on the closed upper half plane.

Corollary 9 motivates the following question: Given $f \in C(\partial\Omega)$ does there exist $u \in C(\overline{\Omega})$ which is harmonic in Ω and agrees with f on $\partial\Omega$?

This is known as the Dirichlet problem.

Remark 11. If the domain Ω does not have a compact closure the solution may not be unique. Consider, for example, $\overline{\Omega} = \{z : \Im z \geq 0\}$ and the function $f(z) = 0$ for real z. Then $u_1(z) = \Im e^{\frac{1}{z}}$ on $\Omega = \{z : \Im z > 0\}$, $u_1(z) = 0$ for real z, and, $u_2(z) = 0$ for all $z \in \overline{\Omega}$, are two distinct functions, both harmonic in the open upper half plane, continuous on the closed upper half plane and which agree with f on the real axis, the boundary of Ω. Indeed a simpler function $v(z) = v(x + iy) = y$ also serves as an illustrative example. It is also a nonconstant (but unbounded) harmonic function on the upper half plane and vanishes on the boundary.

Remark 12. There exist Ω and a continuous bounded real valued function on $\partial\Omega$ such that the associated Dirichlet problem has no solution. Consider $\Omega = \{z : 0 < |z| \leq 1\}$. Then $\partial\Omega = \{|z| = 1\} \cup \{0\}$ Let $f(z) = 0$ if $|z| = 1$ and $f(0) = 1$. The associated Dirichlet problem for this data has no solution. If possible let u be a solution of the Dirichlet problem for boundary value f. Then by the removable singularity theorem (see Thm. 13) u is harmonic in the unit disk. But by the mean value property of harmonic functions $u(0) = 0$ which is a contradiction. Thus the Dirichlet problem has no solution in this case.

Theorem 13. *If u is harmonic and bounded in a deleted neighbourhood of z_0, then $u(z_0)$ can be so defined that the resulting new function u is harmonic in all of the given neighbourhood.*

Proof. Without loss of generality assume that $z_0 = 0$ and $0 \leq |z| < R$ is the neighbourhood of 0 in which u is bounded and harmonic. For the boundary function $u\,|_{\partial B(0,R)}$ we have a harmonic function h on $B(0,R)$ which has a continuous extension up to the boundary and which agrees on the boundary with u. If we can show that $u \leq h$ and $h \leq u$ on $B(0,R) \setminus \{0\}$, it will follow that $u = h$ in $B(0,R) \setminus \{0\}$ and thus u can be defined at 0 to be $h(0)$.

Let $v = (u - h)$ or $v = (h - u)$. Consider, for $\varepsilon > 0$,

$$v_\varepsilon(z) = v(z) + \varepsilon \log(|z|/R), \quad 0 < |z| < R.$$

Then v_ε is harmonic on $0 < |z| < R$ and $v_\varepsilon(z) = 0$ on $|z| = R$. As v is bounded $\limsup_{z\to 0} v_\varepsilon(z) \leq 0$. Hence $v_\varepsilon(z) \leq 0$ on $0 < |z| \leq \delta$ for some δ. Since v_ε is harmonic in the annulus $\delta \leq |z| \leq R$ and non-positive on the boundary of the annulus, we have $v_\varepsilon \leq 0$

in the annulus. Letting $\delta \to 0$ we see that $v_\varepsilon \le 0$ on the punctured disk $0 < |z| < R$. Letting $\varepsilon \to 0$ we see that $v \le 0$. Since v was any of the functions $u - h$ or $h - u$, we see that $h = u$ and the theorem is proved. □

Before going any further let us remark that the concept of harmonic function on any Riemann surface X can be defined in an obvious way:

We say that $u : X \to \mathbb{R}$ is harmonic if for any co-ordinate chart (U, z), the function $f \circ z^{-1} : z(U) \to \mathbb{R}$ is harmonic. This is equivalent to saying that with respect to local co-ordinates $z = (x, y)$, $u_{xx} + u_{yy} = 0$, i.e., u satisfies Laplace equation on X locally. Note that these notions are well-defined by the holomorphic compatibility of the charts.

The maximum principle also continues to be true for harmonic functions on a Riemann surface. For let u be real valued and harmonic in a connected open set Ω of a Riemann surface X. Suppose u attains its maximum at $a \in \Omega$. Let (U, z) be a co-ordinate chart at a with closure lying in Ω. Then $u \circ z^{-1}$ is harmonic in $z(U)$ and attains its maximum at $z(a)$. By maximum principle for harmonic functions on the plane we see that $u \circ z^{-1}$ is constant on $z(U)$, hence u is constant on U. One can now proceed as before and conclude that the set of points on Ω where the maximum is attained is open and closed and since it is nonempty it is all of Ω, Ω being connected. Again, as before, if $\Omega \subset X$ is a connected domain with compact closure, u and v are continuous real valued functions on $\overline{\Omega}$, harmonic in Ω and if $u = v$ on $\partial\Omega$, then $u = v$ on all of $\overline{\Omega}$. We thus see that the Dirichlet problem for a connected domain Ω with compact closure has at most one solution.

2 Dirichlet Problem in the Unit Disk

In spite of the negative results in the last section we shall show in this section that the Dirichlet problem has a solution in any disk. In fact, we establish this for the unit disk. The general case, in fact, the solvability of Dirichlet problem for any domain biholomorphic to the unit disk, follows from this. (Not quite true; see Remark 11 and Remark 19.)

Let $S^1 = \partial\overline{B}(0, 1) = \{z : |z| = 1\}$. Given $f \in C(S^1)$, we wish to find $u \in C(\overline{B}(0, 1))$ such that $\Delta u = 0$ in $B(0, 1)$ and $u\,|_{S^1} = f$.

To motivate the construction of such a u, assume that such a u exists. Then by the mean value property we have

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})d\theta.$$

Now we want to know $u(z)$ for other values of $z \in B(0, 1)$. Consider the autmorphism of the unit disk which takes 0 to z, $Tw = \frac{w+z}{1+\overline{z}w}$. Then T maps $B(0, 1)$ onto $B(0, 1)$ biholomorphically and T is a homeomorphism of S^1. Thus $u \circ T(w)$ is continuous on $\overline{\Omega}$ and harmonic in Ω. Hence, by the mean value property of harmonic functions,

$$u(z) = (u \circ T)(0) = \frac{1}{2\pi} \int_0^{2\pi} f(\frac{e^{i\theta} + z}{1 + \overline{z}e^{i\theta}})d\theta.$$

We make a change of variable $e^{it} = T(e^{i\theta})$ or $e^{i\theta} = T^{-1}(e^{it})$. A calculation shows that

$$d\theta = |(T^{-1})'(e^{it})|dt = \frac{1-|z|^2}{|1-\overline{z}e^{it}|^2}dt = \frac{1-|z|^2}{|e^{it}-z|^2}dt.$$

Thus we get the equation

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\frac{e^{i\theta}+z}{1+\overline{z}e^{i\theta}})d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})K(e^{it},z)dt, \tag{1}$$

called the Poisson Equation, where

$$K(e^{it},z) = \frac{1-|z|^2}{|e^{it}-z|^2}.$$

K is called the Poisson Kernel, more often written as $P(e^{it},z)$ a notation we shall use whenever convenient.

It is easy to see that $K(e^{it},z)$ is harmonic in z being the real part of the function $\frac{e^{it}+z}{e^{it}-z}$ which is holomorphic in z. Now the right hand side of Eq. 1 makes sense whether u in question exists or not. Its equality to the middle term is only an application of the change of variables formula. The middle term is a continuous extension of f inside the unit disk and the right hand side at once shows that the extension is harmonic there, as we see below.

Theorem 14. *Let $f \in C(S^1)$ and define, for $z \in B(0,1)$,*

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\frac{z+e^{it}}{1+\overline{z}e^{it}})dt.$$

Then u is harmonic in $B(0,1)$ and a continuous extension of f into the open unit disk.

Proof. As seen above, the function u is the real part of the function

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\frac{e^{it}+z}{e^{it}-z}dt.$$

Now g is holomorphic in the unit disk as we see by Morera's theorem or by differentiating under the integral sign. Hence u, being the real part of g, is harmonic in the unit disk. To see that u is a continuous extension of f inside the unit disk, we note that $lim_{z\to e^{i\theta}}\frac{z+e^{it}}{1+\overline{z}e^{it}} = e^{i\theta}$ except when $e^{i\theta} = -e^{it}$. Since f is continuous on the compact set S^1, it is bounded on it. By bounded convergence theorem

$$\begin{aligned}\lim_{z\to e^{i\theta}} u(z) &= \frac{1}{2\pi}\int_0^{2\pi} \lim_{z\to e^{i\theta}} f(\frac{z+e^{it}}{1+\overline{z}e^{it}})dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})dt \\ &= f(e^{i\theta}\end{aligned}$$

which shows that u is a continuous extension of f into the open unit disk. □

The above proof of the solution of the Dirichlet problem for the disk seems simpler than the usual proof which uses a further property of the Poisson Kernel, i.e., its being an approximate identity, a property which we explain below.

Various forms of Poisson Kernel

The function

$$K(w,z) = \Re(\frac{w+z}{w-z}) = \frac{|w|^2 - |z|^2}{|w-z|^2},$$

for $w \neq z$, is also called Poisson Kernel. For $|w| = 1$ it takes the form

$$K(e^{i\theta}, z) = \frac{1 - |z|^2}{|e^{i\theta} - z|^2}$$

If we set $w = e^{i\theta}$ and $z = re^{it}, 0 \leq r < 1$ Poisson kernel takes the form

$$K(e^{i\theta}, re^{it}) = \frac{1 - r^2}{1 - 2r cos(\theta - t) + r^2} = \frac{1 - r^2}{|1 - re^{i(\theta - t)}|^2}.$$

When $w = e^{i0} = 1$, K takes the form

$$\begin{aligned} K(1, re^{it}) &= \frac{1 - r^2}{|1 - re^{it}|^2} \\ &= \Re(\frac{1 + re^{it}}{1 - re^{it}}) \\ &= \Re(1 + 2\sum_{n=1}^{\infty} r^n e^{int}) \\ &= \sum_{n=-\infty}^{\infty} r^{|n|} e^{int}. \end{aligned}$$

Poisson Kernel as an Approximate Identity

(1) For $0 \leq |z| < 1$, $K(e^{i\theta}, z) > 0$ since $K(e^{i\theta}, z) = \frac{1 - |z|^2}{|e^{i\theta} - z|^2}$.

(2) We also have

$$\frac{1}{2\pi} \int_0^{2\pi} K(e^{i\theta}, z) d\theta = 1.$$

This follows from the residue theorem since the right hand side of the above equation is equal to

$$\Re[\frac{1}{2\pi} \int_{|w|=1} \frac{w+z}{w-z} \frac{dw}{w}] = 1.$$

This may also be seen using the Fourier expansion

$$K(e^{i\theta}, re^{it}) = \sum_{-\infty}^{\infty} r^{|n|} e^{ni(\theta - t)}.$$

(3) For $0 < \delta < \frac{\pi}{2}$,

$$\sup_{|\theta - t| \geq \delta} K(e^{i\theta}, re^{it}) \to 0 \text{ as } r \to 1.$$

This is immediate since $K(e^{i\theta}, re^{it}) \leq \frac{1 - r^2}{cos^2 \delta/2}$ if $|\theta - t| \geq \delta$.

Ex. 15. Prove Thm. 14 using the above three properties of the Poisson Kernel. *Hint:* Observe that

$$u(z) - \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})K(e^{i\theta}, z)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left(u(z) - f(e^{i\theta})\right) K(e^{i\theta}, z)\, d\theta.$$

Given ε choose δ by using the uniform continuity of f. Split domain of integration into two parts: one over $|\theta| \leq \delta$ and the other over its complement in $[-\pi, \pi]$. Use the third property of the kernel to estimate the integral over $|\theta| > \delta$.

Ex. 16. Let u be a continuous function on an open set Ω which satisfies the mean value property, i.e., for every $a \in \Omega$ and for every r with $\overline{B}(a, r) \subset \Omega$,

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{it})dt.$$

Show that u is harmonic in Ω. *Hint:* Let v be the solution of the Dirichlet problem for the boundary data $u\mid_{\partial B(a,r)}$. Then $v - u$ satisfies the minimum and maximum principles (because of the mean value property).

Ex. 17. Let f be continuous on $\partial B(a, r)$. Show that if $|p - a| < r$ then

$$u(p) = \frac{1}{2\pi}\int_0^{2\pi} \frac{|r|^2 - |p-a|^2}{|a + re^{it} - p|^2} f(a + re^{it})\, dt$$

is the unique continuous and harmonic extension of f in $B(a, r)$.

Ex. 18. Find a "Poisson integral formula" for the solution of the Dirichlet problem in the upper half plane $\{z \in \mathbb{C} : \Im(z) > 0\}$ with a bounded continuous boundary data.

Remark 19. Go through Remark 11 (especially the first example) at the beginning of this section. Try to see why we cannot conclude the uniqueness of the solutions in the upper half plane even though the upper half plane is biholomorphic to the unit disk.

3 Harnack's Inequalities and Harnack's Principle

We have $|w| - |z| \leq |w - z| \leq |w| + |z|$. Using these we obtain estimates for $K(w, z)$ (Recall that $|w| > |z|$.):

$$K(w,z) = \frac{|w|^2 - |z|^2}{|w-z|^2} = \frac{(|w| - |z|)(|w| + |z|)}{|w-z|^2} \leq \frac{(|w-z|)(|w| + |z|)}{|w-z|^2} = \frac{|w| + |z|}{|w-z|} \leq \frac{|w| + |z|}{|w| - |z|}.$$

Also

$$K(w,z) = \frac{|w|^2 - |z|^2}{|w-z|^2} \geq \frac{(|w| - |z|)(|w| + |z|)}{(|w| + |z|)^2} = \frac{|w| - |z|}{|w| + |z|}.$$

Hence

$$\frac{|w| - |z|}{|w| + |z|} \leq K(w,z) \leq \frac{|w| + |z|}{|w| - |z|}. \tag{2}$$

Let us use these inequalities along with the mean value property of harmonic functions. If u is harmonic in $B(a, R^1)$,then for $R < R^1$ we have by the mean value property of harmonic functions:

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + Re^{i\theta})d\theta.$$

Let us assume that u is a nonnegative function, harmonic in $B(a, R^1)$. Let $0 < r < R < R^1$. Then

$$u(a + re^{it}) = \frac{1}{2\pi}\int_0^{2\pi} K(Re^{i\theta}, re^{it})u(a + Re^{i\theta})d\theta.$$

Applying Eq. 2 and the mean value property, we get

$$\frac{R-r}{R+r}u(a) \leq u(a + re^{it}) \leq \frac{R+r}{R-r}u(a).$$

Thus we have obtained:

Theorem 20 (Harnack Inequalities)**.** *If u is continuous and non-negative in $\overline{B}(a, R)$ and harmonic in $B(a, R)$ then for any $z = a + re^{it} \in B(a, R)$*

$$\frac{R-r}{R+r}u(a) \leq u(z) = u(a + re^{it}) \leq \frac{R+r}{R-r}u(a).$$

These inequalities govern the growth of $u(z)$ in a neighbourhood of a.

Theorem 21 (Harnack's Principle)**.** *Let (u_n) be a sequence of real valued harmonic functions on a connected open set Ω in $\mathbb{C}$. Assume that $u_n(z) \leq u_{n+1}(z)$ for all $n \geq 1$ and all $z \in \Omega$. If $(u_n(z))$ is bounded for some $z \in \Omega$, then (u_n) converges uniformly on compact subsets of Ω and the limit function is harmonic in Ω.*

Proof. Let $E = \{a \in \Omega : (u_n(a)) \text{ is bounded.}\}$. By hypothesis E is not empty. We show first that E is both open and closed in Ω.

(1) E is open: Let $a \in E$. Choose $R > 0$ such that $\overline{B}(a, R) \subset \Omega$. Then, by Harnack's inequalities (applied to the nonnegative harmonic function $u_n - u_1$),

$$\frac{R-r}{R+r}(u_n(a) - u_1(a)) \leq u_n(z) - u_1(z) \leq \frac{R+r}{R-r}(u_n(a) - u_1(a))$$

for all $z \in B(a, R)$. Thus $(u_n(z))$ is bounded for all $z \in B(a, R)$ and E is open.

(2) E is closed in Ω: Let $a \in \Omega$ be a cluster point of E. To show that $a \in E$, choose $R > 0$ such that $\overline{B}(a, R) \subset \Omega$. Let $z_0 \in E \cap B(a, R/2)$. By Harnack's inequalities

$$\frac{R - \frac{1}{2}R}{R + \frac{1}{2}R}(u_n(a) - u_1(a)) \leq u_n(z_0) - u_1(z_0) \leq \frac{R + \frac{1}{2}R}{R - \frac{1}{2}R}(u_n(a) - u_1(a)).$$

Since $z_0 \in E$, i.e.,$(u_n(z_0))$ is bounded, we see that $(u_n(a))$ is bounded and $a \in E$. Thus E is closed. Since Ω is connected we conclude that $E = \Omega$.

Thus for any $z \in \Omega$, $(u_n(z))$ is bounded and monotone. Hence it is convergent to a finite value. To prove the uniform convergence of the sequence on compact subsets of Ω it is enough to prove the uniform convergence on a neighbourhood of any $a \in \Omega$. Let $R > 0$ be chosen such that $\overline{B}(a, R) \subset \Omega$. Let $0 < r < R$. Then for all $z \in B(a, r)$

$$\frac{R-r}{R+r}(u_n(a) - u_m(a)) \leq u_n(z) - u_m(z) \leq \frac{R+r}{R-r}(u_n(a) - u_m(a)).$$

Since $(u_n(a))$ converges to a finite real number, we see that (u_n) converges uniformly on $B(a, r)$. Since each u_n satisfies the mean value property (being harmonic) we see that the limit function u, which is continuous, also satisfies the mean value property. By Exer. 16 u is harmonic. This proves the theorem. □

4 Subharmonic Functions and Their Properties

Theorem 22. *Let Ω be a connected open set the plane. Let $v : \Omega \to \mathbb{R}$ be continuous. Then the following are equivalent:*

(1) For every region $D \subset \Omega$ whose closure is compact and contained in Ω and for every function u continuous on $\overline{D}$ and harmonic in D, if $v \leq u$ on ∂D then $v \leq u$ on D (hence also on $\overline{D}$).

(2) For every point $p \in \Omega$ and for every disk $B(p, r)$ whose closure is contained in Ω we have

$$v(p) \leq \frac{1}{2\pi} \int_0^{2\pi} v(p + re^{i\theta})\, d\theta.$$

(3) For every point $p \in \Omega$ and for some disk $B(p, \varepsilon)$ which lies in Ω, for any r, $0 < r < \varepsilon$,

$$v(p) \leq \frac{1}{2\pi} \int_0^{2\pi} v(p + re^{i\theta})\, d\theta.$$

(4) For every region $D \subset \Omega$, for every u harmonic in D satisfying $v \leq u$, either $v < u$ in D or $v = u$ in D.

Proof. (1) implies (2): For let $B(p, r)$ be a disk whose closure lies in Ω and let w be the harmonic function in $B(p, r)$ which agrees with v on the boundary of $B(p, r)$. Then

$$v(p) \leq w(p) = \frac{1}{2\pi} \int_0^{2\pi} v(p + re^{i\theta})\, d\theta$$

where the inequality on the left follows from (1) and the equality on th right is due to the mean value property of the harmonic w.

(2) implies (3) is trivial.

(3) implies (4): Let u be as in (4) and let $w = v - u$. Then $w \leq 0$ in Ω. Assume that $w(a) = 0$ for some $a \in \Omega$. We show that $w = 0$ in all of Ω. The set $E = \{p \in D : w(p) = 0\}$ is clearly non-empty and closed in Ω. We show that E is also open. Let $\varepsilon > 0$ be as in (3). Since u is harmonic in Ω it satisfies the mean value property. In view of (3) we have for any r, $0 < r < \varepsilon$,

$$
\begin{aligned}
w(p) = 0 = v(p) - u(p) &\le \frac{1}{2\pi}\int_0^{2\pi} [v(p+re^{i\theta}) - u(p+re^{i\theta})]\, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} w(p+re^{i\theta})\, d\theta \le 0,
\end{aligned}
$$

since $w \le 0$ in D. Clearly $w(p+re^{i\theta}) = 0$ for all θ and all $r < \varepsilon$, whence $w = 0$ in $B(p,\varepsilon)$ and E is open. Since Ω is connected, we see that $E = \Omega$.

(4) implies (1): Let v, u and D be as in (1). Let $w = v - u$. Then $w \le 0$ on the ∂D. Let $m = max\{w(p) : p \in \overline{D}\}$. Since w is continuous on compact $\overline{D}$, m is attained on this set. If it is attained on the boundary of D then clearly $v \le u$ on D since $v \le u$ on ∂D. Otherwise there is a point $p_0 \in D$ with $w(p_0) = m$. If we replace u by $U = u + m$ we see that U is harmonic in D, $v \le U$ in D and $v(p_0) = U(p_0)$ for $p_0 \in D$. By (4), $v = U$ in all of D, hence on $\overline{D}$. Thus $v - u = m$ on ∂D, and since $v \le u$ on ∂D, $m \le 0$ whence $v \le u$ on D. The theorem is proved. □

Definition 23. Any real valued continuous function u defined on a connected open set Ω is said to be *subharmonic* if it satisfies any one (hence all) of the four conditions of the theorem above.

The following theorem and proposition are immediate consequences of Def. 23 and Thm. 22.

Theorem 24 (Maximum Principle for Subharmonic Functions)**.** *(1) (Weak Form) If v is subharmonic in Ω and $v \le M$ in Ω then either $v < M$ or $v = M$.*

(2) (Strong Form) If $\overline{\Omega}$ is compact, v is continuous on $\overline{\Omega}$ and subharmonic in Ω then v attains its supremum on the boundary of Ω. □

Ex. 25. A continuous function $u\colon \Omega \to \mathbb{R}$ is subharmonic iff for every domain $D \subset \Omega$ and every harmonic function v on Ω, the function $u + v$ has no maximum in D unless $u + v$ is a constant.

Proposition 26. *We have:*

(1) sum of two subharmonic functions is subharmonic, in particular the sum of a harmonic function and a subharmonic function is subharmonic.

(2) nonnegative multiple of a subharmonic function is subharmonic.

(3) $max\{u, v\}$ is subharmonic whenever u and v are subharmonic. □

Poisson Modification

Let v be subharmonic in Ω. Let $D := B(a,r)$ be a disk whose closure lies in Ω. Define P_v on Ω, called the Poisson modification of v, as follows:

$$
P_v(p) = \begin{cases} v(p), & p \in \Omega - B(a,r), \\ \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |a-p|^2}{|a+re^{it}-p|^2} v(a+re^{it})\, dt, & p \in B(a,r). \end{cases}
$$

P_v depends on the disk D. When we wish to emphasize its dependence on D we shall write $P_{D,v}$ or $P_D v$ in place of P_v.

Theorem 27. *Poisson modification of v is subharmonic on Ω.*

Proof. Note that P_v is harmonic in $B(a,r)$ and the values of P_v on the boundary of $B(a,r)$ agree with those of v. (See Exer. 17. It is easy to see that P_v satisfies condition (3) of Thm. 22 on $\Omega - \partial\Omega$. Moreover $v - P_V$ subharmonic on $B(a,r)$, continuous up to the boundary of $B(a,r)$ where it vanishes. So by the strong maximum principle for subharmonic functions we conclude that $v \leq P_v$ on $B(a,r)$. Since $v = P_v$ outside $B(a,r)$, we see that $v \leq P_v$ on Ω. If p is a point in $\partial\Omega$ and $B(p,\rho)$ a disk with centre p and closure lying in Ω, then $v \leq P_v$ on the boundary of this disk and we have:

$$P_v(p) = v(p) \leq \frac{1}{2\pi}\int_0^{2\pi} v(p+\rho e^{it})dt \leq \frac{1}{2\pi}\int_0^{2\pi} P_v(a+\rho e^{it})dt.$$

Thus P_v satisfies condition (3) of Thm. 22 at all points of Ω and P_v is subharmonic in Ω. □

Ex. 28. Show that $u\colon \Omega \to \mathbb{R}$ is subharmonic iff $u \leq P_{D,u}$ for every disk $D \subset \Omega$.

Proposition 29. *Let $u\colon \Omega \to \mathbb{R}$ be C^2. Then u is subharmonic iff $\Delta u := u_{xx} + u_{yy} \geq 0$ on Ω.*

Proof. Since subharmonicity is a local property we may assume, without loss of generality, that u is C^2 and $\Delta u \geq 0$ on $\Omega := B(0,1)$.

First we assume that $\Delta u > 0$. Then u cannot have a maximum in Ω. If false, let $a \in \Omega$ be a point of maximum of u. Then, at a, $\frac{\partial^2 u}{\partial x^2} \leq 0$ and $\frac{\partial^2 u}{\partial y^2} \leq 0$ so that $\Delta u \leq 0$ at this point a — a contradiction.

Now, if h is harmonic in Ω then $\Delta(u+h) > 0$ and hence $u+h$ has no maximum in Ω. Again employing the arguments in the proof of Thm. 22 (or Exer. 25) we see that u is subharmonic in Ω.

To deal with the general case, for any $\varepsilon > 0$ consider the function $v_\varepsilon(x,y) := u(x,y) + \varepsilon(x^2+y^2)$. Then $\Delta v_\varepsilon = \Delta u + 4\varepsilon > 0$. Hence v_ε is subharmonic by the first part. Hence for every disk $D \subset \Omega$ we have $P_{Dv_\varepsilon} \geq v_\varepsilon$. That is, we have

$$P_D u + \varepsilon(x^2+y^2) \geq P_D u + P_D(x^2+y^2) \geq u + \varepsilon(x^2+y^2).$$

Letting $\varepsilon \to 0$ we see that $P_D u \geq u$. Hence by Exer. 28, u is subharmonic.

The other way implication is left as an exercise. □

This Proposition brings out the analogy between the harmonic functions and the "linear" functions (from $\mathbb{R}$ to $\mathbb{R}$) of the form $ax+b$ and that of the subharmonic functions and the (twice differentiable) convex functions characterised by $f'' \geq 0$. It is worthwhile keeping this analogy in mind. In particular, notice that condition 1 of Thm. 22 is the geometric definition of a convex function on $\mathbb{R}$.

5 Subharmonic Functions on a Riemann Surface

Since the property of being harmonic or subharmonic is entirely a local property of a function, it is possible to define such functions on any Riemann surface. We have already defined harmonic functions on a Riemann surface above. A subharmonic function on a Riemann surface is defined as follows.

Let X be a Riemann surface, and Ω a connected open subset of X. A continuous real valued function v on Ω is said to be subharmonic if for every co-ordinate chart (U, z) with $U \subset \Omega$, $v \circ z^{-1}$ is subharmonic in $z(U)$. Let us call a co-ordinate chart (U, z) a parametric disk with centre $p \in U$ and radius r if $z(U)$ is a disk in the complex plane with centre $z(p)$ and radius r. The following theorem lists some useful equivalent definitions of a subharmonic function on a Riemann Surface. The proof is similar to the proof of Thm. 22. We need a notation. Let (U, z) be a parametric disk with centre p and radius r with closure of U lying in Ω. By mean value of v on ∂U, denoted by $M_{v,U}$, we mean

$$M_{v,U} = \frac{1}{2\pi} \int_0^{2\pi} v \circ (z^{-1} \left(z(p) + re^{it}\right))dt.$$

Theorem 30. *Let Ω be a connected open set of a Riemann surface X. Let $v : \Omega \to \mathbb{R}$ be a continuous. Then the following are equivalent:*

(1) For every region $D \subset \Omega$ whose closure is compact and contained in Ω and for every function u continuous on $\overline{D}$ and harmonic in D, if $v \leq u$ on ∂D then $v \leq u$ on D (hence also on $\overline{D}$).

(2) For every point $p \in \Omega$ and for every parametric disk (U, z) with centre p, radius r and whose closure is contained in Ω we have

$$v(p) \leq M_{v,U}.$$

(3) For every point $p \in \Omega$ and for some parametric disk (U, z) with centre p and radius ε which lies in Ω, for any r, $0 < r < \varepsilon$, and $W := z^{-1}(\{\zeta : |\zeta - z(p)| < r\})$, we have

$$v(p) \leq M_{v,W}.$$

(4) For every region $D \subset \Omega$, for every u harmonic in D satisfying $v \leq u$, either $v < u$ in D or $v = u$ in D.

If v is subharmonic in Ω, we can define its Poisson modification as follows: Let (U, z) be a parametric disk with centre p and r whose closure lies in Ω. Let u be the continuous harmonic extension of $v \circ z^{-1}$ from the boundary of $z(U)$ into $z(U)$. We now define the Poisson modification P_v of v as follows:

$$P_{U,v}(q) := P_U v := P_v(q) = \begin{cases} v(q), & q \in \Omega \setminus U; \\ u \circ z(q), & q \in U. \end{cases}$$

As before $v \leq P_v$ and P_v is subharmonic.

Ex. 31. Note that P_v can be defined even for a continuous v which is not subharmonic. Show that v is subharmonic if for all parametric disks (U, z) with closure of U lying in Ω, $v \leq P_{v,U}$.

6 Perron's Method

Theorem 32. *Let Ω be an open connected set in a Riemann Surface X. Let $\mathcal{F}$ be a uniformly bounded non-empty family of subharmonic functions on Ω with the following properties:*
(1) $u, v \in \mathcal{F} \Rightarrow max\{u, v\} \in \mathcal{F}$.
(2) $u \in \mathcal{F} \Rightarrow P_{u,U} \in \mathcal{F}$ for any parametric disk (U, z) whose closure lies in Ω.

Then the function $h(x) = \sup\{u(x) : u \in \mathcal{F}\}$ is harmonic in Ω.

Proof. Let $a \in \Omega$, (U, z) a parametric disk at a whose closure lies in Ω. Choose a sequence $(u_n) \in \mathcal{F}$ such that $u_n(a) \to h(a)$ as $n \to \infty$. If we replace u_n by $v_n = max\{u_1, u_2, \cdots, u_n\}$ and further v_n by $P_{v_n,U}$, in view of hypothesis (1) and (2) of the theorem, we still remain within the family $\mathcal{F}$. Further the resulting sequence still converges, at a, to $h(a)$. Therefore, without loss of generality, we assume that the chosen sequence (u_n) is increasing and that $u_n = P_{u_n,U}$.

Let $u(p) = lim_{n\to}u_n(p), p \in \Omega$. Then $u(a) = h(a)$ and by Harnack's theorem u is harmonic in U. We claim that $u(p) = h(p)$ for all $p \in U$. To see this let $p \in U$ and let (v_n) be an increasing sequence in $\mathcal{F}$ satisfying (i) $u_n \leq v_n = P_{v_n,U}$, (ii) $v_n(p) \to h(p)$. If $v = lim_{n\to\infty}v_n$, then v is harmonic in U by Harnack's theorem. Also $u \leq v$. u and v are harmonic in U with $u(a) = v(a)$. Maximum principle applied to $u - v$ shows that $u = v$ in all U. Thus $u = h$ in U and h is harmonic in U. Since (U, z) is an arbitrary parametric disk in Ω we conclude that h is harmonic in Ω. □

Solution of Dirichlet Problem

Let Ω be a connected open set in a Riemann surface X and $f : \partial\Omega \to \mathbb{R}$ be a bounded continuous function. To avoid triviality we assume that f is non-constant. Let $M = sup\{|f(x)| : x \in \partial\Omega\}$. Let $\mathcal{F}$ denote the set of all (real valued) continuous functions on $\overline{\Omega}$ such that (i) u is subharmonic in Ω and $|u| \leq M$, (ii) $u \mid_{\partial\Omega} \leq f$. Then $\mathcal{F}$ is not an empty family since the constant function $-M$ is in $\mathcal{F}$. Also $\mathcal{F}$ satisfies the conditions of Thm. 32. Hence $h(p) = \sup\{u(p) : u \in \mathcal{F}\}$ for $p \in \Omega$ is harmonic in Ω.

For this h to be a solution of the Dirichlet problem for f we must have $lim_{y\to p}h(y) = f(p)$ for all $p \in \partial\Omega$. Under certain conditions on Ω we can ensure this but not always. Le us call h the Perron function corresponding to the boundary function$f$.

Let us observe, however, that if $m \leq f \leq M$ and u is *any* solution of the Dirichlet problem for this f, then $u = h$. For then, u is harmonic and $m \leq u \leq M$ in Ω by the maximum principle, hence belongs to $\mathcal{F}$. We therefore have $u \leq h$. Let $v \in \mathcal{F}$. Since u is harmonic $v - u$ is subharmonic in Ω and ≤ 0 on $\partial\Omega$. Hence $v - u \leq 0$ on Ω by subharmonicity of $v - u$ or $v \leq u$ in Ω. Clearly $h \leq u$ in Ω. Thus $u = h$.

In the next section we give conditions on $\partial\Omega$ which ensure that the Perron function h is indeed a solution for the Dirichlet problem for the boundary data f.

7 Boundary Behaviour

Definition 33. A point $a \in \partial\Omega$ is called a *peak point* or a *regular point* if there exists an open set V with $a \in V$ and a continuous real valued function P on $\overline{\Omega} \cap V$ with the following properties: (i) P is subharmonic in $\Omega \cap V$, (ii) $P(a) = 0$, (iii) $P(x) < 0$ for all $x \in \overline{\Omega} \cap V \setminus \{a\}$. Such a function P is called a peaking function at a. (In old terminology a peaking function is called a barrier at a.)

We will show presently that if $b \in \partial\Omega$ is a peak point then $lim_{p \to b} h(p) = f(b)$, so that h constructed above has continuous extension to every peak point of $\partial\Omega$, and the value of h at the peak point agrees with the value of f at that point. In particular if every point of $\partial\Omega$ is a peak point then the Dirichlet problem has a solution for any continuous bounded f on $\partial\Omega$. This is the main theorem of this section. A converse of this which is easy to prove is the following:

Theorem 34. *If the Dirichlet problem admits a solution for every bounded continuous f on $\partial\Omega$ then every point of $\partial\Omega$ is a peak point.*

Proof. Let $b \in \partial\Omega$. Choose a continuous f on $\partial\Omega$ such that $-1 \leq f \leq 0$ with $f(p) = 0$ if and only of $p = b$. Let u the solution of the Dirichlet problem for this f. Then $-1 \leq u \leq 0$. By maximum principle $u < 0$ in Ω. Clearly u is a peaking function at b and b is a peak point. □

There are easy geometric conditions which ensure that a boundary point is a peak point. We give two such conditions below:

Condition 1. Let Ω be a connected open set in the complex plane and let $b \in \partial\Omega$. Assume that Ω satisfies the exterior sphere condition at b, i.e.,there exists a disk $B(z_0, r)$ such that $b \in \partial B(z_0, r)$ and $\overline{B}(z_0, r) \cap \Omega = \emptyset$. Then b is a peak point.

Proof. Let $a = \frac{b+z_0}{2}$. Then $P(z) = log\frac{r}{2} - log|z - a|$ is a peaking function at b, hence b is a peak point. □

Condition 2. If $b \in \partial\Omega$ can be reached by an analytic arc (i.e., the image of a line segment under a holomorphic map) which has no points in common with $\overline{\Omega} \setminus \{b\}$,then b is a peak point.

Proof. This is a local problem. We may assume that Ω is a subset of the complex plane and that b can be reached by a line segment with no point in common with $\overline{\Omega}\setminus\{b\}$. We may simplify further by assuming that $b = 0$ and the line segment is $\{(x, y) : y = 0 \ \& \ x \leq 0\}$. Choose a single valued branch of $\sqrt{z}$ in the complement of the negative real axis. Set $-P(z) = \Re(\sqrt{z})$. In polar co-ordinates $-P(z) = -\sqrt{r} cos\theta/2$ with $-\pi \leq \theta \leq \pi$. P peaks at b, and b is therefore peak point of $\partial\Omega$. □

Ex. 35. Show that if $\partial\Omega$ is a C^1 (e.g., given locally as a level surface with nonvanishing gradient) then every point of $\partial\Omega$ is a peak point.

Before we prove the main theorem of this section we need a lemma.

Lemma 36. *Let $a \in \partial\Omega$ be a peak point. Let $m \leq M$ be given. Let U be a neighbourhood at a with compact closure. Then there exists a $u \in C(\overline{\Omega}, \mathbb{R})$ with the following properties:*
u is subharmonic in Ω, $u(a) = M$,
$u(p) \leq M$ on $\overline{\Omega} \cap U$,
$u = m$ on $\overline{\Omega} \setminus U$.

Proof. Let V be a neighbourhood of a which admits a peak function P at a. The number $sup\{P(z) : z \in \overline{\Omega} \cap \partial U\}$ is either not defined (which can happen if the $\overline{\Omega} \cap \partial U$ is empty) or it is less than an negative number $-\delta$ for some $\delta > 0$. Choose N large enough so that $-N\delta < m - M$. Define

$$u(z) = \begin{cases} m, & \text{if } z \in \overline{\Omega} - U \\ M + \max\{m - M, NP(z)\}, & \text{if } z \in \overline{\Omega} \cap U. \end{cases}$$

This u is continuous and subharmonic on $\Omega \cap U$ and equals m in a neighbourhood of $\Omega \setminus (\Omega \cap U)$ and so u is subharmonic in Ω. $\square$

Theorem 37. *Let Ω be a connected open set in a Riemann surface X. Let h be the Perron function associated to a real valued continuous bounded function f on $\partial\Omega$. Then for every peak point $b \in \partial\Omega$ we have*

$$\lim_{y \to b} h(y) = f(b).$$

Proof. Since f is bounded, assume that $m \leq f \leq M$.Let $\varepsilon > 0$ be given. We plan to show that

$$f(b) - \varepsilon \leq \liminf_{y \to b} h(y) \leq \limsup_{y \to b} h(y) \leq f(b) + \varepsilon.$$

For the given $\varepsilon > 0$, we choose neighbourhood V of b with compact closure such that

$$f(b) - \varepsilon \leq f(y) \leq f(b) + \varepsilon,$$

for all $y \in \partial\Omega \cap V$.

Step 1. Using the previous lemma, choose $v \in C(\overline{\Omega})$ with the following properties: (i) it is subharmonic in Ω, (ii) $v(b) = f(b) - \varepsilon$, (iii) $v \leq f(b) - \varepsilon$ on $\overline{\Omega} \cap V$ and (iv) $v = m - \varepsilon$ on $\overline{\Omega} \setminus V$.

For $y \in \partial\Omega \setminus (\overline{\Omega} \cap V)$,

$$v(y) = m - \varepsilon < m \leq f(y).$$

For $y \in \partial\Omega \cap V$,

$$f(b) - \varepsilon \leq f(y) \leq f(b) + \varepsilon.$$

But $v(y) \leq f(b) - \varepsilon$ on $\overline{\Omega} \cap V$. Thus $v \leq f$ on $\partial\Omega$. Thus $v \in \mathcal{F}$. Hence $v \leq h$, or,

$$f(b) - \varepsilon = v(b) = \lim_{y \to b} v(y) \leq \lim \inf_{y \to b} h(y).$$

Step 2. We again use the previous lemma to find a $w \in C(\overline{\Omega})$ such that (i) w is subharmonic in Ω, (ii) $w(b) = -f(b)$, (iii) $w \leq -f(b)$ on $\overline{\Omega} \cap V$ (iv) $w = -M$ on $\overline{\Omega} \setminus V$.

Take any $u \in \mathcal{F}$ and $y \in \partial\Omega \cap V$. Then

$$u(y) \leq f(y) \leq f(b) + \varepsilon.$$

Hence on $\partial\Omega \cap V$,

$$w(y) + u(y) \leq -f(b) + f(b) + \varepsilon = \varepsilon.$$

On $\overline{\Omega} \cap \partial V$,

$$w(y) + u(y) \leq -M + M = 0.$$

Thus the function $w + u$ which is subharmonic in $\Omega \cap V$ satisfies $w + u \leq \varepsilon$ on $\overline{\Omega} \cap V$ by property (1) of Thm. 30. Thus

$$u \leq \varepsilon - w \text{ on } \overline{\Omega} \cap V$$

for all $u \in \mathcal{F}$. Therefore

$$h(y) \leq \varepsilon - w(y)$$

for all $y \in \Omega \cap V$. This implies that

$$\limsup_{y \to b} h(y) \leq \varepsilon - w(b) = \varepsilon + f(b).$$

The theorem is proved. □

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