Double Cosets and Sylow's First Theorem

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Let H be a subgroup of G. Then G acts on G/H transitively. The stabilizer of aH is aHa^{-1}.

Let $K \leq G$. Then we can restrict the action of G on G/H to K. The K-orbit of gH is $\{kgH : k \in K\}$. The stabilizer of gH with K-action is $gHg^{1-} \cap K$. The union of the cosets of H in the K-orbit of gH is KgH. This is called the *double coset* of g relative to K and H.

It is clear that G is the disjoint union of all distinct double cosets relative to K and H.

Let us now assume that G is finite. Let m be the number of elements in the K-orbit of gH. Since the stabilizer of gH in K-action is $gHg^{-1} \cap K$, it follows that m is the index $|K : (gHg^{-1} \cap K)|$. Therefore, the number of elements in the double coset KgH is given by

$$|KgH| = \frac{|K||H|}{|gHg^{-1} \cap K|}. \tag{1}$$

Since $|gHg^{-1} \cap K| = |H \cap g^{-1}Kg|$, we have proved the following result.

Proposition 1. *Let G be finite and R be a complete set of representatives of the distinct double cosets of G relative to K and H. Then*

$$|G| = \sum_{r \in R} \frac{|K||H|}{|H \cap rKr^{-1}|}. \quad \square \tag{2}$$

Example 2. Consider $G = GL(n, \mathbb{Z}_p)$ of invertible $n \times n$ matrices with entries in $\mathbb{Z}_p$. It is easy to see that $|GL(n, \mathbb{Z}_p)| = (p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})$. The highest power of p that divides $|G|$ is therefore $p^{1+2+\cdots+n} = p^{\frac{n(n-1)}{2}}$. The subgroup of unimodular upper triangular matrices Details!

$$P := \left\{ \begin{pmatrix} 1 & & & \star \\ & 1 & & \\ & & \ddots & \\ & 0 & & 1 \end{pmatrix} \right\} \leq GL(n, \mathbb{Z}_p)$$

is of order $p^{\frac{n(n-1)}{2}}$ and hence it is the Sylow p-subgroup of G.

Lemma 3. *Let $H \leq G$. Let P be Sylow p-subgroup of G. Then there exists $g \in G$ such that $H \cap g^{-1}Pg$ is a Sylow p-subgroup of H.*

Proof. We exploit (2). Consider the double coset decomposition of G relative to H and P. We then have

$$|G| = \sum_{r\in R} \frac{|H||P|}{|H \cap rPr^{-1}|}.$$

We observe that both sides are divisible by $p^n := |P|$. If we cancel p^n from both sides, then at least one of the terms on the right side (after the division), say, $\frac{|H|}{|H \cap rPr^{-1}|}$ is not divisible by p. This term is the index $|H : H \cap rPr^{-1}|$. It follows that $H \cap rPr^{-1}$ is p-group and its index is not divisible by p. We therefore conclude that it a Sylow p-subgroup of H. □

Theorem 4 (Sylow's Theorem-1)**.** *Let G be be a finite group. Then G contains a Sylow p-subgroup.*

Proof. Let $|G| = n$. By left action, we have a one-one homomorphism of G into S_n. This is Cayley's theorem. We now embed S_n in $GL(n, \mathbb{Z}_p)$ as follows. For each $\sigma \in S_n$, let us consider the permutation matrix, a matrix $(e_{\sigma(1)}, \dots, e_{\sigma(n)})$ got from the identity matrix $(e_1, e_2, \dots e_n)$. Thus G is isomorphic to a subgroup H of $GL(n, \mathbb{Z}_p)$. Let P be the Sylow p-subgroup of unimodular upper triangular matrices in $GL(n, \mathbb{Z}_p)$. Then by the last lemma, there exists a conjugate Q of P such that $Q \cap H$ is a Sylow p-subgroup of H. Its pre-image under the isomorphism $G \to H$ will be a Sylow p-subgroup of G. □