Double Series

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Definition 1. Let $a\colon \mathbb{N}\times\mathbb{N}\to\mathbb{C}$ be a map. Then we say that a is a double sequence and we usually let $a_{ij}=a(i,j)$ and $(a_{ij})=a$. We say that (a_{ij}) is convergent if there exists a complex α with the property: if $\varepsilon>0$ is given, there exists $N\in\mathbb{N}$ such that $|a_{mn}-\alpha|<\varepsilon$ for all $m,n\geq N$. We then write $\lim_{m,n\to\infty}a_{mn}=\alpha$.

Ex. 2. Prove that (a) $\lim_{m,n}\frac{1}{m+n}=0$ and (b) $(\frac{m}{m+n})$ is not convergent.

Theorem 3. *Assume that* $\lim_{m,n}z_{mn}$ *exists and that for each* m, $\lim_{n\to\infty}z_{mn}$ *exists. Then,* $\lim_{m\to\infty}(\lim_{n\to\infty}z_{mn})$ *exists and is equal to* $\lim_{m,n\to\infty}z_{mn}$.

Proof. Let $\lim_{m,n}z_{mn}=z$ and $\lim_{n\to\infty}z_{mn}=z_m$ for $m\in\mathbb{N}$. Given $\varepsilon>0$, choose $N\in\mathbb{N}$ such that

$$|z_{mn}-z|<\varepsilon \text{ for } m,n\geq N. \tag{1}$$

Also, for given m, there exists $N_m\in\mathbb{N}$ such that

$$|z_{mn}-z_m|<\varepsilon \text{ for } n\geq N_m. \tag{2}$$

We claim that $|z_m-z|<2\varepsilon$ for $m\geq N$. For choose $n>\max N,N_m$. The claim now follows from Eq. 1 and Eq. 2. □

Ex. 4. Show that $\lim_{m,n}\frac{(-1)^n}{m}$ exists but not $\lim_{n\to\infty}\frac{(-1)^n}{m}$ for any m.

Corollary 5. *Assume that (1) for each* m, $\lim_{n\to\infty}z_{mn}$ *exists, (2) for each* n, $\lim_{m\to\infty}z_{mn}$ *exists and (3)* $\lim_{m,n}z_{mn}$ *exists. Then* $\lim_{m\to\infty}(\lim_{n\to\infty}z_{mn})$ *and* $\lim_{n\to\infty}(\lim_{m\to\infty}z_{mn})$ *exist and are equal.* □

Ex. 6. The double sequence in Ex. 2.b shows that

$$\lim_{m\to\infty}(\lim_{n\to\infty}z_{mn}) \text{ and } \lim_{n\to\infty}(\lim_{m\to\infty}z_{mn}) \text{ exist and are not equal.}$$

Ex. 7. Let (x_{mn}) be bounded. Assume that it is an increasing sequence of each of its variable. Then it is convergent.

Theorem 8. *Assume that each term* x_{mn} *of a double sequence is real. Assume that, for each* m, $(x_{mn})_{n=1}^{\infty}$ *is an increasing sequence of* n *and that, for each* n, $(x_{mn})_{m=1}^{\infty}$ *is an increasing sequence of* m. *If one of the limits* $\lim_{m,n\to\infty}x_{mn}$, $\lim_{m\to\infty}(\lim_{n\to\infty}x_{mn})$, $\lim_{n\to\infty}(\lim_{m\to\infty}x_{mn})$ *exists, then the other two limits also exist and all three are equal.*

Analogous result holds if the word increasing is substituted by the word decreasing.

Proof. Let $\lim_{m,n} x_{mn} = x$. Then $x_{mn} \leq x$ for all m, n. (Why?) Hence, for each m, $(x_{mn})_{n=1}^{\infty}$ is a bounded increasing sequence and so $\lim_{n\to\infty} x_{mn}$ exists. Similarly, $\lim_{m\to\infty} x_{mn}$ exists. Therefore, by the corollary, the other two iterated limits exist and are equal.

Let $\lim_{m\to\infty}(\lim_{n\to\infty} x_{mn})$ exist. Let $\lim_{n\to\infty} x_{mn} = y_m$ for $m \in \mathbb{N}$. Let $\lim_m y_m = y$. Since x_{mn} increases with n, $x_{mn} \leq y_m$ for all n. Since x_{mn} increases with m, (y_m) is an increasing sequence of m so that $y_m \leq y$ for all m. Thus, we have

$$x_{mn} \leq y_m \leq y \qquad \text{for all } m, n.$$

Then (x_{mn}) converges by Ex. 7. By the first part, $\lim_n(\lim_m x_{mn})$ exists and all three limits are equal. □

Definition 9. Let (z_{mn}) be a double sequence. Let $s_{mn} := \sum_{i=1}^{m}\sum_{j=1}^{n} z_{ij}$. If the double sequence (s_{mn}) converges to s, we say that the double series $\sum_{m,n} z_{mn}$ converges and its sum is s.

Lemma 10. *If $\sum_{m,n} a_{mn}$ converges to s and if $\sum_n a_{mn}$ converges for each m, then $\sum_m(\sum_n a_{mn})$ converges to s.*

Proof. Follows from Th. 3. □

Ex. 11. Let $\sum_{m,n} a_{mn}$ and $\sum_{m,n} b_{mn}$ be double series of positive terms. Assume that $a_{mn} \leq b_{mn}$ and that $\sum_{m,n} b_{mn}$ converges. Then $\sum_{m,n} a_{mn}$ converges.

Theorem 12. *Let (a_{mn}) be a double sequence in $\mathbb{C}$. If any one of the series $\sum_{m,n} |a_{mn}|$, $\sum_m(\sum_n |a_{mn}|)$, $\sum_n(\sum_m |a_{mn}|)$ converges, then all the series $\sum_{m,n} a_{mn}$, $\sum_m\sum_n a_{mn}$, $\sum_n\sum_m a_{mn}$ converge and all have the same sum.*

Proof. We first prove the result when the double sequence is real. Let

$$b_{mn} := (|a_{mn}| + a_{mn})/2 \quad \text{and } c_{mn} := (|a_{mn}| - a_{mn})/2.$$

Then $b_{mn} \geq 0$ and $c_{mn} \geq 0$, $a_{mn} = b_{mn} - c_{mn}$ and

$$0 \leq b_{mn}, c_{mn} \leq |a_{mn}|.$$

If one of the three series converges, so does the corresponding series with $|a_{mn}|$ replaced by b_{mn} by the comparison test in Ex. 11. Then, by Th. 8, $\sum_{m,n} b_{mn}$, $\sum_m\sum_n b_{mn}$ and $\sum_n\sum_n b_{mn}$ all converge to the same sum. Similar result holds for the series involving c_{mn}'s. Taking differences of the b and c series yield the result for the real case.

If a_m are complex, we let $a_{mn} := x_{mn} + iy_{mn}$. Since $|x_{mn}|, |y_{mn}| \leq |a_{mn}|$, if one of the three a series converge, then the corresponding x and y series both converge. □

Ex. 13. Find, when they exist, the double limit $\lim_{m,n\to\infty} x_{mn}$ and the iterated limits $\lim_m \lim_n x_{mn}$ and $\lim_n \lim_m x_{mn}$ of the sequence whose (m,n)-th term is (i) $\frac{m-n}{m+n}$, (ii) $\frac{m+n}{m^2}$, (iii) $\frac{m+n}{m^2+n^2}$ and (iv) $\frac{mn}{m^2+n^2}$.

Ex. 14. Does there exist an unbounded convergent double sequence?

Ex. 15. Is there an analogue of the absolute convergence implies convergence principle for double series?