Regularity Theorem for Elliptic Operators

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

1 Introduction

Abstract

This article has some very annoying typographical mistakes which may lead you to think they are mathematical in nature! Beware! I did not proof-read them carefully. I would appreciate anybody who send me the list of corrections. This exposition is accessible to all those who need the result in their domain but are not experts in analysis.

Let Ω be an open set in $\mathbb{R}^n$ and L be a linear partial differential operator on Ω. That is,

$$L = \sum_{|\alpha| \le k} a_\alpha D^\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in \mathbb{Z}^+$, $D^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n}$, $a_\alpha \in C^\infty(\Omega)$. We say that the order of L is k if $a_\alpha \neq 0$ for some α with $|\alpha| = k$.

The main question of these lectures is the following question: Consider an equation of the type $Lu = f$ in Ω. Assume $u \in C^k(\Omega)$ and $f \in C^r(\Omega)$. Then is $u \in C^{k+r}(\Omega)$?

Regularity theorem essentially says that under certain conditions on L, the answer is in the affirmative. (Not quite in these words — however for the time being this will do!)

We are already familiar with many cases where such a thing happens:

(1) Let $f\colon \mathbb{R} \to \mathbb{R}$ be C^k. Let $u\colon \mathbb{R} \to \mathbb{R}$ be a C^1-solution of the equation $\frac{du}{dx} = f$. Then u is in fact C^{k+1}.

(2) $\Omega = \mathbb{R}^2$ and $L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. $u \in C^2(\Omega)$ is said to be harmonic if $Lu = 0$. Here 0 is a C^∞ function and so we would expect that if u satisfies $Lu = 0$, then $u \in C^\infty(\mathbb{R}^2)$. Indeed this is true since if v is a harmonic conjugate of u, then $f = u + iv$ is an analytic function of a complex variable and hence infinitely differentiable. So its real part, which is u, is infinitely differentiable.

(3) Let $\Omega = \mathbb{C}$ and let $\frac{\partial u}{\partial \bar{z}} = 0$, $u \in C^1(\Omega)$. Then u is analytic and hence C^∞.

But the following example shows that for regularity theorem we need some conditions on L.

(4) Let $u \in C^2(\mathbb{R})$. Consider the wave operator $L = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$. By the transformation $X = x + y, \quad Y = x - y$, the equation $(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2})u = 0$ is transformed into the equation $\frac{\partial^2}{\partial X \partial Y} u = 0$. Now if we take $f(X, Y) = h(X)$, where $h(X) \in C^2$ but not in C^3, then $\frac{\partial^2}{\partial X \partial Y} f = 0$, but f is not in C^3.

2 Weak Derivatives

Let $\Omega \subset \mathbb{R}^n$. Let L be differential operator of order k and let $u \in C^k(\Omega)$, $f \in C^0(\Omega)$ such that $Lu = f$. Also let $\phi \in C_c^\infty(\Omega)$. Then $\langle Lu, \phi \rangle = \langle f, \phi \rangle$, where $\langle h, g \rangle = \int h(x) g(x)\, dx$. For a moment let us suppose that $L = \frac{\partial}{\partial x_1}$. Then $\langle Lu, \phi \rangle = \langle f, \phi \rangle$ means

$$\int \frac{\partial u}{\partial x_1} \phi(x)\, dx = \int f(x) \phi(x)\, dx \tag{1}$$

Integrate the left hand side of Eq. 1 in parts. The first term (the boundary term) is zero because $\phi \in C_0^\infty(\Omega)$ and Ω is open. So we get

$$-\int u \frac{\partial \phi}{\partial x_1}\, dx = \int f(x) \phi(x)\, dx$$

Similarly, if $L = D^\alpha$ we get

$$(-1)^{|\alpha|} \int u D^\alpha \phi\, dx = \int f(x) \phi(x)\, dx \tag{2}$$

Conversely, let Eq. 2 be satisfied for all $\phi \in C_c^\infty(\Omega)$. Then $\langle D^\alpha u, \phi \rangle = \langle f, \phi \rangle$ for all $\phi \in C_c^\infty(\Omega)$. That is, $\langle D^\alpha u - f, \phi \rangle = 0$ for all $\phi \in C_c^\infty(\Omega)$. One then easily shows that $D^\alpha u = f$. (See exercise below).

Ex. 1. Let $\psi \in C(\Omega)$ such that $\int_\Omega \psi \phi = 0$ for all $\phi \in C_c^\infty(\Omega)$. Then show that $\psi = 0$.

Now we make the following definition:

Definition 2. Let $u \in L^1_{loc}(\Omega)$ (i.e., for all $K \subset \Omega$ K compact, we have $\int_K |u| < \infty$). We say that $D^\alpha u = f$ in Ω, *in the weak sense* if $(-1)^{|\alpha|} \int u D^\alpha \phi = \int f(x) \phi(x)\, dx$ for all $\phi \in C_c^\infty(\Omega)$.

Note that in the case where $u \in C^{|\alpha|}(\Omega)$, $D^\alpha u = f$ in the weak sense iff $D^\alpha u = f$ in the calculus or classical sense.

Example 3. Let $u \colon \mathbb{R} \to \mathbb{R}$ be defined by $u(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 1 & \text{if } 1 < x < 2 \end{cases}$. We claim that $v := \begin{cases} 1 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } 0 < x < 2 \end{cases}$ is the weak derivative of u. To prove this, we need to show that

$$\int_0^2 u \varphi' = -\int_0^2 v \varphi, \qquad \text{for } \varphi \in C_c^\infty((0, 2)).$$

We split the integral on the left side into two:

$$\begin{aligned}\int_0^2 u\varphi' &= \int_0^1 x\varphi' + \int_1^2 \varphi' \\ &= [x\varphi(x)]_0^1 - \int_0^1 \varphi + \varphi(2) - \varphi(1) \\ &= \varphi(1) - \int_0^2 \varphi(x) - \varphi(1), \quad \text{since } \varphi(2) = 0 \\ &= -\int_0^2 v\varphi.\end{aligned}$$

The simplicity of the above argument should not lull the reader into false confidence. So, let us look at another example.

Example 4. Let $u\colon \mathbb{R} \to \mathbb{R}$ be defined by $u(x) = \begin{cases} x & \text{if } 0 < x \leq 1 \\ 2 & \text{if } 1 < x < 2 \end{cases}$. You may be tempted to guess a weak derivative of u . We claim that u has no weak derivatives. Assume that v is a weak derivative of u. Hence we have

$$\int_0^2 u\varphi' = -\int_0^2 v\varphi, \qquad \text{for } \varphi \in C_c^\infty((0,2)). \tag{3}$$

Let us work on the left side:

$$\begin{aligned}\int_0^2 u\varphi' &= \int_0^1 x\varphi' + \int_1^2 2\varphi' \\ &= [x\varphi]_0^1 - \int_0^1 \varphi + 2(\varphi(2) - \varphi(1)) \\ &= -\varphi(1) - \int_0^1 \varphi + 2\varphi(2) - 2\varphi(1) \\ &= \varphi(2) - \varphi(1) - \int_0^1 \varphi \\ &= -\varphi(1) - \int_0^1 \varphi. \end{aligned} \tag{4}$$

From Eq. 3 and Eqellip:eq4, it follows that v must satisfy the equation

$$\int_0^2 v\varphi = \varphi(1) + \int_0^1, \qquad \text{for all } \varphi \in C_c^\infty(0,2). \tag{5}$$

We choose φ_n such that $\varphi(1) = 1$ and support of φ_n shrinks to the point 1. Then we have

$$1 + \int_0^1 \varphi_n = \int_0^2 v\varphi_n.$$

Using the dominated convergence theorem, we get $1 = 0$. This contradiction proves our claim.

For every L there exists a unique differential operator L^* (called the *formal adjoint* of L) defined by the equation $\langle Lu, \phi\rangle = \langle u, L^*\phi\rangle$ for all u, $\phi \in C_c^\infty(\Omega)$. If $L = \sum a_\alpha(x) D^\alpha$ then L^* is given by $L^*u = \sum (-1)^{|\alpha|} D^\alpha(\overline{a_\alpha(x)})u(x)$.

Definition 5. Let $u \in L^1_{loc}(\Omega)$ and $f \in C(\Omega)$. u is said to be a *generalised solution* or a *weak solution* of $Lu = f$ on Ω if and only if $\langle u, L^*\phi\rangle = \langle f, \phi\rangle$ for all $\phi \in C_c^\infty(\Omega)$.

If $u \in C^k(\Omega)$ (where k is the order of L) then it is easily seen that u is a generalised solution of $Lu = f$ if and only if u is a classical solution.

3 Distributions and their Derivatives

Definition 6. A distribution u on Ω is a linear map $u : C_c^\infty(\Omega) \to \mathbb{C}$ such that given a compact subset $K \subset \Omega$ there exist $k \in \mathbb{Z}^+$ and $C_K > 0$ such that

$$|u(\phi)| \leq C_K \sum_{|\alpha|\leq k} \sup_K |D^\alpha \phi| \quad \text{for all } \phi \in C_c^\infty(\Omega).$$

Note that for each k and K, $p_{k,K}(\phi) = \sum_{|\alpha|\leq k} \sup_K |D^\alpha\phi|$ defines a seminorm on $C_c^\infty(\Omega)$ and the above condition means that u is continuous with respect to the family of seminorms $p_{k,K}$.

Example 7. Let $f \in L^1_{loc}(\Omega)$. Define $u_f(\phi) = \int f(x)\phi(x)\,dx$ for $\phi \in C_c^\infty(\Omega)$. Then u_f is a distribution with $k = 0$ in the definition. We say that u_f is represented by f.

Example 8. Let μ be a regular Borel measure. Define $u(\phi) = \int \phi\,d\mu$. Then u is a distribution with $k = 0$.

Example 9. Assume that $0 \in \Omega$. Define $\delta(\phi) = \phi(0)$. Then δ is a distribution with $k = 0$. But δ cannot be represented by any f i.e., $\delta \neq u_f$ for any $f \in L^1_{loc}(\Omega)$. δ is called the Dirac distribution.

Notation. We denote the space of distributions over Ω by $D'(\Omega)$. Example 1 shows that $D'(\Omega) \supset L^1_{loc}(\Omega)$. If u is a distribution and $\phi \in C_c^\infty(\Omega)$ we denote $u(\phi)$ by (u, ϕ).

Definition 10. The i-th partial derivative $\frac{\partial u}{\partial x_i}$ of a distribution u is the distribution defined by $(\frac{\partial u}{\partial x_i}, \phi) := -(u, \frac{\partial \phi}{\partial x_i})$.

The higher derivatives are defined in an analogous way by the equation $(D^\alpha u, \phi) := (-1)^{|\alpha|}(u, D^\alpha\phi)$.

Ex. 11. Show that $\frac{\partial u}{\partial x_i}$ is a distribution. More generally, show that $D^\alpha u$ is a distribution.

Ex. 12. Let $u = u_f$, where $f \in C^k(\Omega)$. Then $D^\alpha f$ makes sense for all α with $|\alpha| \leq k$ and $\int D^\alpha f\phi = (-1)^{|\alpha|}\int uD^\alpha\phi$, by integration by parts, since f has compact support. This shows that $D^\alpha u_f = u_{D^\alpha f}$.

Ex. 13. Define the Heaveside function $H \in L^1_{loc}(\mathbb{R})$ by $H(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$. Then $\frac{d}{dx}u_H$ is the Dirac distribution δ.

Definition 14. Let L be a linear partial differential operator on Ω. We say that $Lu = f$ on Ω in the distribution sense if $(u, L^*\phi) = \langle f, \phi\rangle$ for all $\phi \in C_c^\infty(\Omega)$.

Ex. 15. If $u = u_g$ for $g \in L^1_{loc}$, then $Lu = f$ in the distribution sense if and only if $Lg = f$ in the weak sense.

4 Fourier Series

Consider the set $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. The set $\{\chi_n = e^{in\theta},\, n \in \mathbb{Z},\, 0 \leq \theta \leq 2\pi\}$ is a complete orthonormal basis for $L^2(S^1)$ which is nothing other than the $L^2([0, 2\pi])$ the space of square integrable periodic functions with period 2π. If $f \in L^2(S^1)$, f has the Fourier series representation $f = \sum (f, \chi_n)\chi_n$. Here the series converges in the L^2 sense. We also have the Parseval relation $\| f \|^2_{L^2(S^1)} = \sum |\widehat{f}(n)|^2$, where $\widehat{f}(n) = \langle f, \chi_n \rangle_{L^2}$.

The following observation is the cornerstone for our analysis of the problem.

Lemma 16. *Assume $f \in C^1(S^1)$. The partial sums of the Fourier series of f converges to f uniformly on S^1, i.e., in the supremum norm on $C(S^1)$.*

Proof. Let $s_n(f) := \sum_{|k| \leq n} f_n e^{-inx}$ denote the n-th partial sum of the Fourier series. Now we claim that the Fourier series for $f' \in C^0(S^1) \subset L^2$ is given by $f' = \sum \widehat{f'}(n) e^{inx}$ where $\widehat{f'}(n) = -in\widehat{f}(n)$. For, observe that we have, by an integration by parts,

$$\widehat{f'}(n) := \frac{1}{2\pi} \int_0^{2\pi} f'(t) e^{-int}\, dt = \frac{1}{2\pi} \left[f(t) e^{-int} \,|_0^{2\pi} - \int_0^{2\pi} f(t) e^{-int}\, dt \right] = in\widehat{f}(n).$$

Hence

$$\begin{aligned} \sum_{n \neq 0} |\widehat{f}(n)| &= \sum_{n \neq 0} \frac{|\widehat{f'}(n)|}{|n|} \\ &\leq (\sum_{n \neq 0} |\widehat{f'}(n)|^2)^{\frac{1}{2}} (\sum_{n \neq 0} \frac{1}{n^2})^{\frac{1}{2}} \\ &< \infty \text{ since } f' \in C(S^1) \subseteq L^2(S^1). \end{aligned}$$

This implies that the series $\sum \widehat{f}(n) e^{inx}$ is uniformly convergent by Weierstrass M-test. Hence $s_n(f)$ tends to a continuous function g as $n \to \infty$ in the supremum norm. But the supremum norm is stronger than the L^2-norm: $\| f \|^2_L \leq \sqrt{2\pi} \| f \|_\infty$. Hence $s_n(f) \to g$ in the L^2-norm. But already $s_n(f) \to f$ in the L^2-norm. Therefore $f = g$ a.e. But f and g are continuous. Therefore $f \equiv g$. □

Ex. 17. Let $f \in C^\infty(S^1)$. Show that if $f = \sum \widehat{f}(n) e^{inx}$ is the Fourier series representation of f, then $D^k f$ is given by $\sum \widehat{f}(n)(in)^k e^{inx}$. This series converges uniformly to $D^k f$.

Remark 18. We have $\widehat{f'}(n) = in\widehat{f}(n)$. Now $\sum |\widehat{f'}(n)|^2 < \infty$ by Parseval relation. Hence $|n\widehat{f}(n)| \to 0$ as $n \to \infty$. Or in Landau's notation $\widehat{f}(n) = o(\frac{1}{|n|})$. Thus the regularity of f is reflected in the behaviour of the Fourier coefficients $\widehat{f}(n)$ as $|n| \to \infty$. More generally, if $f \in C^k(S^1)$, then $\widehat{f}(n) = o(\frac{1}{|n|^k})$. This remark is crucial and lays the foundation for our approach to the question raised in the introduction.

Now let us consider the r-dimensional torus $\mathbb{T} = S^1 \times \ldots \times S^1$ (r times). We adopt the following notation: $x := (x_1, \ldots, x_r)$, $n = (n_1, \ldots, n_r) \in \mathbb{Z}^r$, $n \cdot x = n_1 x_1 + \ldots + n_r x_r$,

$\chi_n(x) := e^{in\cdot x}$ and $f_n := \langle f, \chi_n\rangle$, the L^2-inner product of the functions. Let $f \in L^2(\mathbb{T})$. Then f has the Fourier series representation

$$f = \sum_{n\in\mathbb{Z}^r} f_n e^{inx} \qquad \text{in } L^2(\mathbb{T}).$$

The following exercise, though trivial, is repeatedly used in the sequel, often without explicit mention.

Ex. 19. Let $f\colon \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ be defined by $f(x) := \|x\|^\alpha$ for $\alpha \in \mathbb{R}$. Then $f \in L^1(\mathbb{R}^n\backslash B(0,\varepsilon))$ for any $\varepsilon > 0$ iff $\alpha < -n$. *Hint:* Use the spherical polar coordinates and the change of variable formula.

Ex. 20. Let $k > r/2$. If $f \in C^k(\mathbb{T})$, then as in the one dimensional case show that the Fourier series of f converges uniformly to f. *Hint:* Last exercise may be of use.

Ex. 21. If $f \in C^\infty(\mathbb{T})$ and if $f(x) = \sum_{n\in\mathbb{Z}^r} f_n e^{in\cdot x}$ then $D^\alpha f$ has as its Fourier series $\sum f_n (in)^\alpha e^{inx}$, where $(in)^\alpha = (in_1)^{\alpha_1}\cdots(in_r)^{\alpha_r}$.

5 The Laplacian Δ and Sobolev Spaces

The Laplacian $\Delta = \sum_{i=1}^r \frac{\partial^2}{\partial x_i{}^2}$ maps $C^2(\mathbb{T})$ to $C(\mathbb{T})$. We would like to extend this to an operator on $L^2(\mathbb{T})$. We would like to define $\Delta : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ as follows: Let $f \in L^2(\mathbb{T})$, $f = \sum f_n e^{inx}$. Define $\Delta f = \sum f_n(-|n|^2)e^{inx}$. But there is one problem — this series may not converge in the L^2 norm. For example take $f = \sum_{n\in\mathbb{Z}\backslash\{0\}} \frac{1}{n} e^{inx}$. On the other hand, we can define $\Delta : C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ by $\Delta f = \sum f_n(-|n|^2)e^{inx}$ and since $C^\infty(\mathbb{T})$ is dense in $L^2(\mathbb{T})$ we can ask whether Δ can be extended to the whole of $L^2(\mathbb{T})$ continuously. But this is not possible either. For $\Delta : C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ is not continuous, as the bounded set $\{e^{inx} \mid n \in \mathbb{Z}^r\}$ in $L^2(\mathbb{T})$ is taken to the unbounded set $\{-|n|^2 e^{inx} \mid n \in \mathbb{Z}^r\}$ in $L^2(\mathbb{T})$.

So in order to define Δ in a meaningful way, we proceed as follows. Consider the collection of formal series $H = \{\sum_{n\in\mathbb{Z}^r} v_n e^{inx}\}$. Fix $s \in \mathbb{Z}$. Look at those v_n's for which $\sum |n|^{2s}|v_n|^2 < \infty$. We have a natural seminorm on this space given by $\sum v_n e^{inx} \mapsto (\sum |n|^{2s}|v_n|^2)^{\frac{1}{2}}$. We modify this seminorm to a norm $\sum((1+|n|^2)^s|v_n|^2)^{\frac{1}{2}}$. We call this normed linear space the s-th Sobolev space and denote it by H_s.

Definition 22. Let $s \in \mathbb{Z}$. The s-th Sobolev space H_s is defined by

$$\begin{aligned} H_s &= \{v = \sum_{\text{(formal sum)}} v_n e^{inx} \mid \|v\|_s^2 := \sum_{n\in\mathbb{Z}^r} (1+|n|^2)^s |v_n|^2 < \infty\}, \\ &= \{v = (v_n)_{n\in\mathbb{Z}^r} \mid \|v\|_s^2 := \sum_{n\in\mathbb{Z}^r} (1+|n|^2)^s |v_n|^2 < \infty\}. \end{aligned}$$

Remark 23. H_s is a Hilbert space. If $u, v \in H_s$ where $u = \sum u_n e^{inx}$, $v = \sum v_n e^{inx}$, we define

$$\langle u, v\rangle_s = \sum u_n \overline{v_n} (1+|n|^2)^s \tag{6}$$

$\langle u, v \rangle_s$ makes sense since

$$\begin{aligned} |\langle u, v \rangle_s| &\leq \sum |u_n|(1+|n|^2)^{\frac{s}{2}} |v_n|(1+|n|^2)^{\frac{s}{2}} \\ &\leq (\sum |u_n|^2(1+|n|^2)^s)^{\frac{1}{2}} (\sum |v_n|^2(1+|n|^2)^s)^{\frac{1}{2}} \text{ (by Cauchy – Schwartz)} \\ &= \|u\|_s \|v\|_s \\ &< \infty. \end{aligned}$$

Ex. 24. Show that $C^\infty(\mathbb{T}) \subset H_s$. Prove that H_s is a complete metric space. In fact H_s is the completion of $(C^\infty(\mathbb{T}), \| \|_s)$.

Ex. 25. Show that $H_t \subset H_s$ for $s \leq t$.

Can we give some concrete representation for H_s? We can identify $H_0(\mathbb{T})$ with $L^2(\mathbb{T})$. For $s \geq 0$, $H_s \subseteq H_0 = L^2$. Also, if $s > 0$, then $H_s \subset H_0$.

Ex. 26. $\sum_{n\in\mathbb{Z}} \frac{1}{n} e^{inx}$ is in $H_0(S^1)$ but is not in any H_s for $s > 0$.

Lemma 27. *H_{-s} is the isometric dual of H_s for all $s \in \mathbb{Z}$.*

Proof. If $u \in H_{-s}$ and $v \in H_s$, define a functional $H_s \to \mathbb{C}$ by $v \to \langle v, u \rangle = \sum v_n \overline{u}_n$. $\langle v, u \rangle$ is defined since

$$\begin{aligned} |\sum v_n \overline{u}_n| &\leq \sum |v_n||u_n|(1+|n|^2)^{\frac{s}{2}}(1+|n|^2)^{-\frac{s}{2}} \\ &\leq (\sum |v_n|^2(1+|n|^2)^s)^{\frac{1}{2}} (\sum |u_n|^2(1+|n|^2)^{-s})^{\frac{1}{2}} \\ &= \|v\|_s \|u\|_{-s} \\ &< \infty. \end{aligned}$$

The rest of the proof is left as an exercise. □

Definition 28. Recall that when $s \geq 0$, $H_s \subseteq H_0 = L^2$. For $f \in H_s$, $f = \sum f_n e^{inx}$, we define $D^\alpha f$ formally by $D^\alpha f = \sum f_n (in)^\alpha e^{inx}$, where $\alpha = (\alpha_1, \ldots, \alpha_r)$, α_i non-negative. Then f has "L^2-derivatives" of order less than or equal to s, i.e., for $|\alpha| \leq s$, $D^\alpha f$ defines an element of L^2. For,

$$\begin{aligned} \sum |(in)^\alpha|^2 |f_n|^2 &= \sum |n_1|^{2\alpha_1} \ldots |n_r|^{2\alpha_r} |f_n|^2 \\ &\leq \sum ({n_1}^2 + \ldots + {n_r}^2)^{|\alpha|} |f_n|^2 \\ &\ll \sum (1+|n|^2)^{|\alpha|} |f_n|^2 \\ &< \infty \text{ if } |\alpha| \leq s. \end{aligned}$$

Now since the formal derivative $D^\alpha f \in L^2$, it is in L^1_{loc} and hence one would expect that it is the α^{th} derivative of f in the weak sense also. Indeed, this is true. All we have to prove is that

$$\int D^\alpha f \phi \, dx = (-1)^{|\alpha|} \int f D^\alpha \phi \, dx \text{ for all } \phi \in C^\infty(\mathbb{T}).$$

That is, we have to prove that

$$\int f_n (in)^\alpha e^{inx} \phi \, dx = (-1)^{|\alpha|} \int f D^\alpha \phi \, dx$$

(from the continuity of the L^2 inner product). But this follows by integration by parts.

Lemma 29. $D^\alpha \colon H_s \to H_{s-|\alpha|}$ *is continuous.*

Proof. Let $u = \sum u_n e^{inx}$, $D^\alpha u = \sum u_n (in)^\alpha e^{inx}$. Let us compute

$$\begin{aligned} \| D^\alpha u \|_t^2 &= \sum |u_n|^2 |n|^{2\alpha} (1+|n|^2)^t \\ &\ll \sum |u_n|^2 (1+|n|^2)^{|\alpha|+t} \end{aligned}$$

If $|\alpha| + t \leq s$, i.e., $t \leq s - |\alpha|$, then this series converges. Hence D^α maps H_s continuously into $H_{s-|\alpha|}$. □

Remark 30. If $u = \sum u_n e^{inx} \in H_s$ we can define Δu formally, where Δ is the Laplacian. Then

$$(1-\Delta)^t u = \sum_n (1+|n|^2)^t e^{inx}$$

Now, which space does $(1-\Delta)^t u$ lie in? Let $(1-\Delta)^t u \in H_k$. $u \in H_s$. So $(1-\Delta)^t u \in H_k$, i.e., $\sum |u_n|^2 (1+|n|^2)^{2t+k} < \infty$ if $2t + k \leq s$. So $(1-\Delta)^s$ maps H_s into H_{-s}. This map is invertible and the inverse is given by $(1-\Delta)^{-s}$.

Lemma 31. *The operator* $(1-\Delta)^s \colon H_s \to H_{-s}$ *is an isometry.*

Proof. Let $u = \sum u_n e^{inx} \in H_s$. Then

$$\begin{aligned} \| (1-\Delta)^s u \|_{-s}^2 &= \sum |u_n|^2 (1+|n|^2)^{2s} (1+|n|^2)^{-s} \\ &= \sum |u_n|^2 (1+|n|^2)^s \\ &= \| u \|_s^2. \end{aligned}$$

□

Ex. 32. Show that for $u, v \in H_s$, we have

$$\langle u, v \rangle_s = \langle (1-\Delta)^s u, v \rangle_0 = \langle u, (1-\Delta)^s v \rangle_0.$$

Remark 33. $\Delta^s : H_s \to H_{-s}$ cannot be invertible since it has got non-trivial kernel. In fact $\Delta^s u = 0$ for any constant u.

If $s \geq 0$, we have seen that $H_s \subset L^2$ and if $t \leq s$ then $H_s \subseteq H_t$. Now we may ask the following question: For what value of $s > 0$ is it true that $H_s \subset C(\mathbb{T})$? Let $\sum u_n e^{inx} \in H_s$. Now $\sum u_n e^{inx} \in C(\mathbb{T})$ if the series $\sum u_n e^{inx}$ converges uniformly. This is true if $\sum |u_n|$ converges. Now

$$\begin{aligned} \sum |u_n| &= \sum |u_n| (1+|n|^2)^{\frac{s}{2}} (1+|n|^2)^{\frac{-s}{2}} \\ &\leq \Big(\sum |u_n|^2 (1+|n|^2)^s\Big)^{\frac{1}{2}} \Big(\sum \frac{1}{(1+|n|^2)^s}\Big)^{\frac{1}{2}} \\ &= \| u \|_s \Big(\sum \frac{1}{(1+|n|^2)^s}\Big)^{\frac{1}{2}} \end{aligned}$$

$\sum \frac{1}{(1+|n|^2)^s} \ll \sum \frac{1}{|n|^{2s}}$ converges if $2s > r$, i.e., if $s > \frac{r}{2}$. Thus $\sum |u_n|$ converges if $s > \frac{r}{2}$. So an element $\sum u_n e^{inx}$ of H_s is in $C(\mathbb{T})$ if $s > \frac{r}{2}$ where r is the dimension of T.

Now let us see when $u \in H_s$ is in $C^1(\mathbb{T})$. Let $u = \sum u_n e^{inx} \in H_s$. Formally differentiating once with respect to, say, x_1, we get

$$D_1 u = \sum u_n (in_1) e^{inx}. \tag{7}$$

Let us see when the above series is uniformly convergent. It is so if the series $\sum |u_n||n_1|$ is convergent. Now,

$$\begin{aligned}
\sum |u_n||n_1| &\leq \sum |u_n|(1+|n|^2)^{\frac{1}{2}} \\
&= \sum |u_n|(1+|n|^2)^{\frac{s}{2}}(1+|n|^2)^{-(\frac{s}{2}-\frac{1}{2})} \\
&\leq (\sum |u_n|^2(1+|n|^2)^s)^{\frac{1}{2}}(\sum \frac{1}{(1+|n|^2)^{s-1}})^{\frac{1}{2}} \\
&= \| u \|_s (\sum \frac{1}{(1+|n|^2)^{s-1}})^{\frac{1}{2}} \\
&\ll \| u \|_s (\sum \frac{1}{|n|^{2s-2}})^{\frac{1}{2}} \\
&< \infty \quad \text{if } 2s-2 > r, \text{ i.e., if } s > \frac{r}{2}+1.
\end{aligned}$$

Thus if $s > \frac{r}{2}+1$ then the series Eq. 7 is uniformly convergent and hence the series $\sum u_n e^{inx}$ which is got by integrating Eq. 7 is in C^1. Iterating the above process we get

Lemma 34 (Sobolev Lemma)**.** *If $s > \frac{r}{2}+k$, then $H_s \subseteq C^k(\mathbb{T})$.* □

Corollary 35. *$\bigcap_{s\in\mathbb{Z}} H_s = C^\infty(\mathbb{T})$.* □

It is very useful to have an equivalent norm on H_s:

Definition 36. Let $u \in H_s$ where $s > \frac{r}{2}+k$. Then by the Sobolev lemma $u \in C^k(\mathbb{T})$. We define a new norm $|\ |_k$ on H_s by

$$|u|_k = \sum_{|\alpha|\leq k} \| D^\alpha u(x) \|_{L^2}.$$

Lemma 37. *If $u \in H_s$, $s > r/2+k$, then $\| u \|_k \ll |u|_k$.*

Proof. We observe that

$$\| D^\alpha u \|_0^2 = \sum |n|^{2\alpha}|u_n|^2 \leq \sum |u_n|^2(1+|n|^2)^k.$$

To get the other way we note that both $(1+|x|^2)^{\frac{k}{2}}$ and $\sum_{\alpha\leq k}|x^\alpha|$ are nonzero for nonzero x and grow like $|x|^k$ at infinity so that we have

$$(1+|x|^2)^{\frac{k}{2}} \leq C \sum_{\alpha\leq k} |x^\alpha|,$$

for some $C > 0$. □

6 Distributions on T

Definition 38. A distribution on $\mathbb{T}$ is a functional $u : C^\infty(\mathbb{T}) \to \mathbb{C}$ such that there exists $k \in \mathbb{Z}^+ \cup \{0\}$, $C \in \mathbb{R}^+$ such that

$$|u(\phi)| \le C \sum_{|\alpha| \le k} \sup_{x \in \mathbb{T}} |D^\alpha \phi(x)|$$

We denote the space of distributions on $\mathbb{T}$ by $D'(\mathbb{T})$.

Proposition 39.

$$D'(\mathbb{T}) = \bigcup_{s \in \mathbb{Z}} H_s.$$

Proof. Let $u \in D'(\mathbb{T})$. Then there exists $k \in \mathbb{Z}^+ \cup \{0\}$, $c \in \mathbb{R}^+$ such that

$$\begin{aligned}
|u(\phi)| &\le C \sum_{|\alpha| \le k} \sup_{x \in \mathbb{T}} |D^\alpha \phi(x)|, \qquad \phi \in C^\infty(\mathbb{T}) \\
&\le C \sum_{|\alpha| \le k} \sum_n |\phi_n| |n|^{|\alpha|} \\
&\le C' \sum_n |\phi_n| |n|^k \\
&= C' \sum_n |\phi_n| |n|^{k + [\frac{r}{2}]\frac{1}{2} + \frac{1}{2}} \cdot |n|^{-[\frac{r}{2}]\frac{1}{2} - \frac{1}{2}} \\
&\le C' (\sum_n |\phi_n|^2 |n|^{2k + [\frac{r}{2}] + 1})^{\frac{1}{2}} (\sum_{|n| \ne 0} \frac{1}{|n|^{[\frac{r}{2}]+1}})^{\frac{1}{2}} \\
&\le C'' (\sum_n |\phi_n|^2 (1 + |n|^2)^{k + [\frac{r}{2}] + 1})^{\frac{1}{2}} \\
&= C'' \|\phi\|_s \text{ where } s = k + \frac{r}{2} + 1.
\end{aligned}$$

This shows that $u : C^\infty \hookrightarrow H_s \to \mathbb{C}$ is continuous. But $C^\infty(\mathbb{T})$ is dense in H_s. So we can extend u to a continuous functional on H_s. So we can consider u as an element of H_{-s}. Thus we have proved that $D'(\mathbb{T}) \subseteq \cup_{s \in \mathbb{Z}} H_s$.

Conversely, let $u \in H_s$ for some s. If $s \ge 0$ then $u \in H_0 \subset L^2(\mathbb{T})$, hence locally L^1 and hence defines a distribution. So let us consider the case where $u \in H_{-s}$, $s > 0$, i.e., $u \in H_s^*$, $s > 0$. Let $\phi \in C^\infty(\mathbb{T}) \subseteq H_s$, where $\phi = \sum \phi_n e^{inx}$. Also, let $u = \sum u_n e^{inx}$.

$$\begin{aligned}
|u(\phi)| &= |\sum \phi_n \overline{u}_n| \\
&\le (\sum |u_n| (1 + |n|^2)^{\frac{-s}{2}} \cdot |\phi_n| (1 + |n|^2)^{\frac{s}{2}}) \\
&\le (\sum |u_n|^2 (1 + |n|^2)^{-s})^{\frac{1}{2}} (\sum |\phi_n|^2 (1 + |n|^2)^s)^{\frac{1}{2}} \\
&= \|u\|_{-s} \|\phi\|_s \\
&\ll \|u\|_{-s} (\sum_{|\alpha| \le s} \|D^\alpha \phi\|_{L^2}) \\
&\le \|u\|_{-s} \cdot c \cdot \sum_{|\alpha| \le s} \sup |D^\alpha \phi|.
\end{aligned}$$

Thus u is a distribution. $\square$

Remark 40. For distributions we have already defined distributional derivatives. Also, for $u \in L^1_{loc}$, we defined weak derivatives and for $u \in H_s$ we defined formal derivative. Now if $u \in C^k(\mathbb{T})$ we have the usual derivative. We also showed that these definitions coincide.

Now we have identified $D'(\mathbb{T})$ with $\cup_{s\in\mathbb{Z}} H_s$. We have to show that under this identification both definitions of derivatives, viz., formal and distributional, coincide.

Suppose $u \in H_t$, where $u = \sum u_n e^{inx}$ and let $\phi \in C^\infty(\mathbb{T})$ and $\phi = \sum \phi_n e^{inx}$. We have the formal derivative of u given by $D^\alpha_F(u) = \sum u_n (in)^\alpha e^{inx}$ and also the distributional derivative of u given by $(D^\alpha_d u, \phi) = (u, (-1)^{|\alpha|} D^\alpha(\phi))$. Now,

$$\begin{aligned}(D^\alpha_F(u), \phi) &= \sum \overline{u_n (in)^\alpha} \phi_n \\ &= (-1)^{|\alpha|} \sum \overline{u}_n (in)^\alpha \phi_n\end{aligned}$$

while

$$\begin{aligned}(D^\alpha_d u, \phi) &= (-1)^{|\alpha|}(u, D^\alpha \phi) \\ &= (-1)^{|\alpha|} \sum \overline{u}_n (in)^\alpha \phi_n\end{aligned}$$

and hence $D^\alpha_F u = D^\alpha_d u$. $\square$

Theorem 41. *Let L be a linear partial differential operator $L = \sum a_\alpha D^\alpha$, where $\alpha \in C^\infty(\mathbb{T})$. Then $L\colon H_s \to H_{s-k}$ is continuous.*

Proof. As we have earlier proved that $D^\alpha\colon H_s \to H_{s-|\alpha|}$ $(\alpha > 0)$ is continuous, the result follows from the next lemma. $\square$

Lemma 42. *Let $\phi \in C^\infty(\mathbb{T})$. Let $\Phi(u) := \phi u$ for $u \in H_s$. Then there exists $C > 0$ such that $\|\Phi(u)\|_s \le C \|u\|_s$ for $u \in C^\infty(\mathbb{T})$.*

Proof. One may perhaps multiply the Fourier series of ϕ and u and then try to estimate $\|\phi u\|_s$. But this is an extremely tedious process. So instead we make use of the fact that for $u \in H_k$, $k \ge 0$,

$$\|u\|_k \ll \sum_{|\alpha| \le k} \|D^\alpha u\|_0.$$

So we have to prove that for $s \ge 0$,

$$\sum_{|\alpha| \le s} \|D^\alpha(\phi u)\|_0 \le c(\phi) \sum_{|\alpha| \le s} \|D^\alpha u\|_0.$$

Fix an α with $|\alpha| = s$. Since $s \ge 0$ and ϕ, $u \in C^\infty$, we can differentiate using Leibniz rule and so

$$\begin{aligned}\|D^\alpha(\phi u)\|_0 &\le \|\phi D^\alpha u\|_0 + \sum_{|\gamma|<s} \left\| D^{\mathbf{b}} \phi D^\gamma u \right\| \\ &\le \|\phi\|_\infty \|D^\alpha u\|_0 + \sum_{|\gamma|<s} c_i \|D^\gamma u\|_0\end{aligned}$$

Similarly, we do for each γ such that $|\gamma| < s$ and get

$$\sum_{|\alpha|<s} \| D^{\alpha}(\phi u) \|_0 \leq c(\phi) \sum_{|\alpha|\leq s} c_i \| D^{\alpha} u \|_0$$

Thus $s \geq 0$, $\phi : C^{\infty}(\mathbb{T}) \subseteq H_s \to H_s$ is continuous and so we can extend ϕ to a continuous operator $\tilde{\phi} : H_s \to H_s$.

Now let us consider the case when s is negative. We want to prove that $\phi : C^{\infty}(\mathbb{T}) \subseteq H_{-s} \to H_{-s}$ $(s > 0)$ is continuous, i.e., we want to prove that $\| \phi u \|_{-s} \leq c(\phi) \| u \|_{-s} \forall u \in C^{\infty}(\mathbb{T}) \subseteq H_{-s}$. Now, H_{-s} is the isometric dual of H_s. So it is enough to prove that $|(\phi u, w)| \leq c(\phi) \| u \|_{-s} \| w \|_s$ $\forall w \in H_s$. Since $C^{\infty}(\mathbb{T})$ is dense in H_s we need to prove it for $w \in C^{\infty}(\mathbb{T})$ only. Now,

$$\begin{aligned}
|(\phi u, w)| &= |\sum \overline{(\phi u)}_n w_n| \\
&= |\int w \overline{\phi u}| \\
&= |\int (w\overline{\phi})\overline{u}| \\
&= |\sum (w\overline{\phi})_n (1+|n|^2)^{\frac{s}{2}} (\overline{u}_n (1+|n|^2))^{\frac{s}{2}}| \\
&\leq \| \overline{u} \|_{-s} \| \overline{\phi} w \|_s \\
&= \| u \|_s \| \overline{\phi} w \|_s \\
&\leq \| u \|_s c(\overline{\phi}) \| w \|_s \quad \text{(by case 1, since } s > 0\text{)}.
\end{aligned}$$

Thus $\phi : C^{\infty}(\mathbb{T}) \hookrightarrow H_{-s} \to H_{-s}$ is continuous. But $C^{\infty}(\mathbb{T})$ is dense in H_{-s}. So we can extend ϕ to a continuous operator on H_{-s}. □

7 Elliptic Operators

Let $L = \sum a_{\alpha} D^{\alpha}$, where $a_{\alpha} \in C^{\infty}(\mathbb{T})$ and the order of L is k. We have seen that L takes H_s into H_{s-k} and $L : H_s \to H_{s-k}$ is continuous. Now one may naturally ask the following question: Let $u \in H_t$ for some t and $Lu = f$, where $f \in H_s$. Then does this imply that $u \in H_{s+k}$?

The answer is in the affirmative if one is able to show that $\| u \|_{s+k} \ll \| Lu \|_s$. But this is not possible. For example, this is not true for constant functions. So let us modify this and ask whether

$$\| u \|_{s+k} \leq C(\| Lu \|_s + \| u \|_t) \tag{8}$$

can be true, where C is some constant depending on s, k and t. We shall see that under certain conditions on L, Eq. 8 holds. Then Eq. 8 is called Garding's inequality.

Now let us see what should be the condition imposed on L such that Eq. 8 holds. Let us consider the case of a homogeneous differential operator with constant coefficients, i.e.,$L = \sum_{|\alpha|=k} a_{\alpha} D^{\alpha}$, where $a_{\alpha} \in \mathbb{C}$.

Definition 43 (Symbol of L)**.** The symbol $\sigma(L)$ of L is defined as

$$\sigma(L) = \sum_{|\alpha|=k} a_\alpha n^\alpha.$$

We then have

$$Lu = \sum u_n \sigma(L)(n)(i)^{|\alpha|} e^{inx}.$$

$Lu \in H_s$ implies that

$$\| Lu \|_s^2 = \sum |u_n|^2 |\sigma(L)(n)|^2 (1 + |n|^2)^s < \infty.$$

Now if $|\sigma(L)(n)|^2 \ll (1+|n|^2)^k$, then we see that $\| u \|_{s+k} \leq c \| Lu \|_s$, where $c = c(s,k)$. But this may not be possible. First, we note that if $n \in \mathbb{Z}^r$ is such that $\sigma(L)(n) = 0$ then $\sigma(L)(l \cdot n) = 0$ for all $l \in \mathbb{Z}$. Hence $|\sigma(L)(n)|^2 \ll (1 + |n|^2)^k$ for $n \ll 0$ only if $\sigma(L)(n) \neq 0$ for any non-zero $n \in \mathbb{Z}^r$. Thus we are led to require that $\sigma(L)(\xi) \neq 0$ for $\xi \in \mathbb{R}^r \setminus \{0\}$. This condition is also sufficient in the sense that if $\sigma(L)(\xi) \neq 0$ for $\xi \neq 0$ then $|\sigma(L)(\xi)|^2 \ll (1 + |\xi|^2)^k$ since we can find constants c_1 and c_2 such that

$$c_1 |\sigma(L)(\xi)|^2 \leq (1 + |\xi|^2)^k \leq c_2 |\sigma(L)(\xi)|^2.$$

Note however that $\sigma(L)(0) = 0$, since L is homogeneous. Hence if $u_n = 0$ for all $n \neq 0$, then $Lu = 0$ and we can never get $\| u \|_{s+k} \leq c \| Lu \|_s$. So we need an error term to take care of this situation. What we have done above shows that if $\sigma(L)(\xi) \neq 0$ for $\xi \in \mathbb{R}^n \setminus \{0\}$ then there exists a constant c such that

$$\| u \|_{s+k} \leq c(\| Lu \|_s + \| u \|_t)$$

Now we make the definition of ellipticity.

Definition 44. Let L be a homogeneous differential operator of order k with constant coefficients. L is said to be *elliptic* if $\sigma(L)(\xi) \neq 0$ for $\xi \in \mathbb{R}^n \setminus \{0\}$.

More generally, let $L = \sum_{|\alpha| \leq k} a_\alpha D^\alpha$; $a_\alpha \in C_\infty(\mathbb{T})$. For $x \in \mathbb{T}$, we define the symbol $\sigma(L)(x, \cdot)$ by $\sigma(L)(x, \xi) = \sum_{|\alpha|=k} a_\alpha(x) \xi^\alpha$. $(\sigma(L)(x, .) : \mathbb{R}^n \to \mathbb{R}^n)$.

We define L to be elliptic at a point x if $\sigma(L)(x, \xi) \neq 0$ for $\xi \in \mathbb{R}^n \setminus \{0\}$. L is said to be *elliptic* if it is elliptic at each point $x \in \mathbb{T}$.

Thus if L is an elliptic homogeneous partial differential operator of order k, then $\| u \|_{s+k} \leq c((\| Lu \|_s + \| u \|_t)$.

The basic inequality concerning the linear elliptic partial diferential operators is

Theorem 45 (Garding's Inequality)**.** *Let L be elliptic of order $k \geq 1$. Let $u \in H_t$ for some t and $Lu \in H_s$. Then we have*

$$\| u \|_{s+k} \leq c(\| Lu \|_s + \| u \|_t),$$

where c is a constant depending only on s, k and t. □

Remark 46. If $t \geq s + k$ then the inequality is trivially true. We postpone the proof of this inequality. Assuming this we derive the consequences.

8 Regularity on the Torus

Lemma 47. **[Rellich's Lemma]** *Let $s < t$. The natural inclusion $H_t \hookrightarrow H_s$ is compact. That is, the image of the closed unit ball in H_t is totally bounded in H_s.*

Proof. Let $u := \sum u_n e^{inx}$ be in the unit ball of H_t. Write $u = \sum_{|n|\le N} u_n e^{inx} + \sum_{|n|>N} u_n e^{inx}$ for an N to be chosen later. We compute the norm of the second term in H_s:

$$\begin{aligned}\sum_{|n|>N} |u_n|^2(1+|n|^2)^s &\le \sum_{|n|>N} |u_n|^2(1+|n|^2)^t(1+|n|^2)^{s-t} \\ &\le \sum_{|n|>N} |u_n|^2(1+|n|^2)^t(1+|N|^2)^{s-t} \\ &< \varepsilon,\end{aligned}$$

if N is chosen sufficiently large.

The first term lies in a finite dimensional space of v's such that if $v_n \neq 0$ then $|n| \le N$. Since any finite dimensional normed linear space is locally compact, we can find an ε-net for such a set. Thus the full unit ball in H_t is totally bounded in H_s. □

In the course of the proof we have proved the following

Corollary 48. *Let $t < s$. Given $\varepsilon > 0$, there exists $N_0 \gg 0$ such that if $u \in H_s$ with $u_n = 0$ for all $|n| \le N_0$, then $\| u \|_t \le \varepsilon \| u \|_s$.* □

Proposition 49 (Fredohlm Property)**.** *Let L be an elliptic operator of order k with C_∞ coefficients so that $L : H_k \subseteq H_0 \to H_0$. Then* $\ker L$ *and* coker L *are finite dimensional.*

Proof. We use the corollary to prove that $\ker L$ is finite dimensional. For that, for $n \in \mathbb{N}$, define $H_s(N) = \{u \in H_s \mid u_n = 0 \text{ for } |n| \le N\}$. We have $L : H_s(N) \to H_{s-k}$, with L elliptic operator of order k. By Garding's inequality,

$$\| u \|_s \le c(\| Lu \|_{s-k} + \| u \|_t) \qquad \text{for } u \in H_t.$$

Now let $u \in H_s(N)$. We can also consider $u \in H_t$ for any $t < s$. So fix some $t < s$. Given $\varepsilon > 0$, we can choose $N \gg 0$ such that for $u \in H_s(N)$, $\| u \|_t \le \varepsilon \| u \|_s$ (by the above corollary). Therefore

$$\| u \|_s \le c\| Lu \|_{s-k} + c\varepsilon \| u \|_s.$$

Now choose ε such that $c\varepsilon \le \frac{1}{2}$ and choose N accordingly. Hence $\| u \|_s \le c\| Lu \|_{s-k}$. This means that L is a topological isomorphism of $H_s(N) \to LH_s(N) \subseteq H_{s-k}$. Now if Y, Z, M are linear spaces and $T : Y \oplus Z \to M$ is a homomorphism such that $T \mid_Y$ is one-one, then $\dim\ker T \le \dim Z$. Therefore $\dim\ker L \le \dim H_s(N)^\perp$ is finite. □

Alternate Proof. Let $L : H_k \to H_0$. Suppose $\ker L$ is not finite dimensional. Then there exists a countable collection of linearly independent vectors $\{u^l\}$ in $H_k \subseteq H_0$ such that

$L(u^l) = 0$ for all l. We can find an orthonormal family $\{e^l\}_{l\in\mathbb{N}}$ in H_0 and $L(e^l) = 0$ for all $l \in \mathbb{N}$. By Garding's inequality,

$$\left\| e^l \right\|_k \leq c(\left\| Le^l \right\|_0 + \left\| e^l \right\|_0) = c$$

That is, $\{\left\| e^l \right\|_k\}_{l\in\mathbb{N}}$ is bounded. Since $H_k \hookrightarrow H_0$ is a compact operator, this means that $\{e^l\}_{l\in\mathbb{N}}$ has a convergent subsequence in H_0. But this is absurd since $\left\| e^l - e^m \right\|_0 = \sqrt{2}$ for $l \neq m$. Thus $\ker L$ is finite dimensional.

coker L is isomorphic to $\ker L^*$. Now if L is elliptic then we can show that L^* is elliptic. Therefore $\ker L^*$ is finite dimensional and hence coker L is finite dimensional. □

We now use these results to prove the regularity of solutions of elliptic partial differential operators on $\mathbb{T}$. We shall indicate two proofs of this result.

Theorem 50 (Regularity on $\mathbb{T}$)**.** *Let L be an elliptic partail differential operator on $\mathbb{T}$ of order k. Let $f \in H_s(\mathbb{T})$. Let $u \in H_t$ for some t be such that $Lu = f$. Then $u \in H_{s+k}$.*

Proof. We may assume that $t < s + k$. Let $H_s(N)$ denote the space of $v \in H_s$ such that $v_n = 0$ for $|n| > N$. For $N \gg 0$ we have

$$\|\varphi\|_{-s} \ll \|L^*\varphi\|_{-s-k}, \qquad \varphi \in H_s(N).$$

We now compute

$$\begin{aligned} \langle u, L^*\varphi\rangle &= \langle f, \varphi\rangle \\ &\leq \|f\|_s \|\varphi\|_{-s} \\ &\ll \|f\|_s \|L^*\varphi\|_{-s-k}. \end{aligned}$$

Thus u defines a bounded linear map on $L^*C^\infty(\mathbb{T}) \subset H_{-s-k}$. The closure of $L^*C^\infty(\mathbb{T})$ in H_{-s-k} is of finite codimension. Thus the space generated by it and a finite number of elements of H_{-s-k} is dense in H_{-s-k}. Thus $u \in H^*_{-s-k}$, the dual of H_{-s-k}. That is to say $u \in H_{s+k}$. □

Proof 2. Write $u = \sum u_n e^{inx}$. Let $f_N = \sum_{|n|\leq N} f_n e^{inx}$ for $N \in \mathbb{N}$. Note that $Pu_N = f_N$. (Verify this.) Using Garding's inequality for the C^∞ function $u_N - u_M$, we get

$$\|u_N - u_M\|_{s+d} \leq C(\|f_N - f_M\|_s + \|u_n - u_M\|_t).$$

As $u_N \to u$ in H_t and $f_n \to f$ in H_s, we deduce that (u_N) is Cauchy in H_{s+d}. Since H_{s+d} is complete, $u_N \to \tilde{u}$ in H_{s+d}. We already have $u_N \to u$ in H_{s+d}. Hence, by uniqueness of limits, we conclude that $\tilde{u} = u$. In particular, $u \in H_{s+d}$. □

9 Garding's Inequality

We now prove Garding's inequality for the general case.

Theorem 51. *Let L be an elliptic operator of order k, $L = \sum_{|\alpha|\leq k} a_\alpha D^\alpha$, $a_\alpha \in C^\infty(\mathbb{T})$. If $u \in H_t$ and $Lu \in H_s$, then there exists a constant c which depend only on L, s and t such that $\|u\|_{s+k} \leq c(\|Lu\|_s + \|u\|_t)$.*

Proof. **Case 1:** $L = \sum_{|\alpha|=k} a_\alpha D^\alpha$, $a_\alpha \in \mathbb{C}$.

We have already proved this case.

Case 2: *Step 1.* Let $L = \sum_{|\alpha|=k} a_\alpha D^\alpha$, where $a_\alpha \in C^\infty(\mathbb{T})$. Let $x_0 \in \mathbb{T}$ and $L_0 = \sum_{|\alpha|=k} a_\alpha(x_0) D^\alpha$. By **Case 1**, for any $u \in H_t$ such that $Lu \in H_s$, we have

$$\|u\|_{s+k} \leq c(\|L_0 u\|_s + \|u\|_t) \tag{9}$$

Now recall that if $P = \sum_{|k|\leq l} b_\alpha D^\alpha$, then $P : H_{s+l} \to H_s$, P is continuous and $\|Pu\|_s \leq c_1(b_\alpha \mid |\alpha| = k)\|u\|_{s+l} + c_2(b_{\mathbf{b}} \mid |\mathbf{b}| \leq k-1)\|u\|_{s+l-1}$ for any $u \in H_{s+l}$, where the constants c_1 and c_2 can be made small if $\sup |b_\alpha|$'s are sufficiently small. Therefore Eq. 9 becomes

$$\begin{aligned} \|u\|_{s+k} &\leq c(\|L_0 u\|_s + \|u\|_t) \\ &\leq c(\|(L_0 - L)u\|_s + \|Lu\|_s + \|u\|_t) \\ &\leq c(c_1\|u\|_{s+k} + c_2\|u\|_{s+k-1} + \|Lu\|_s + \|u\|_t) \end{aligned}$$

If $\sup |a_\alpha(x) - a_\alpha(x_0)| < \varepsilon$, where ε is chosen small enough so that $cc_1 \leq \frac{1}{2}$ then $\|u\|_{s+k} \leq c(c'\|u\|_{s+k-1} + \|Lu\|_s + \|u\|_t)$.

Case 2': Proceeding exactly as in **Case 2** we get $\|u\|_{s+k} \leq c(c'\|u\|_{s+k-1} + \|Lu\|_s + \|u\|_t)$.

Step 2 Since a_α's are uniformly continuous on $\mathbb{T}$, given $\varepsilon > 0$, there exists $\delta = \delta(L)$, such that the coefficients of L oscillates within ε on any ball of radius δ. So take a δ corresponding to the ε given in **Case 2** and cover $\mathbb{T}$ with δ balls. Since $\mathbb{T}$ is compact we can extract a subcover. Choose a partition of unity $\{\phi_i\}_{i=1}^N$ subordinate to the above covering. Let $u \in H_t$ for some t. Then $u = \sum \phi_i u$ since $\sum \phi_i = 1$. Therefore $\|u\|_{s+k} \leq \sum \|\phi_i u\|_{s+k}$. Since $\operatorname{supp} \phi_i$ is contained in a δ-ball and $\sup |a_\alpha(x) - a_\alpha(x')| < \varepsilon$ for x, x' in a δ-ball from **Step 1** we get that

$$\|\phi_i u\|_{s+k} \leq (c_i'\|\phi_i u\|_{s+k-1} + \|L\phi_i u\|_s + \|\phi_i u\|_t).$$

Now

$$\|L(\phi_i u)\|_s \leq \|\phi_i L(u)\|_s + c(\phi_i)\|u\|_{s+k-1}.$$

Therefore

$$\|u\|_{s+k} \leq C(c'\|u\|_{s+k-1} + \|Lu\|_s + \|u\|_t). \tag{10}$$

Now Garding's inequality is trivial for $t \geq s+k$. So we assume that $t \leq s+k-1$. Then $\|u\|_t \leq \|u\|_{s+k-1} \leq \|u\|_{s+k}$. Now suppose that given any $\varepsilon > 0$ we can find a $c(\varepsilon)$ such that

$$\|u\|_{s+k-1} \leq \varepsilon\|u\|_{s+k} + C(\varepsilon)\|u\|_t \tag{11}$$

Then if we choose ε such that $Cc'\varepsilon \leq \frac{1}{2}$, we get Garding's inequality from Eq. 11. So let us prove Eq. 11.

We want to show that for $r \leq s \leq t$, $\|u\|_s^2 \leq \varepsilon\|u\|_t^2 + C(\varepsilon)\|u\|_r^2$. That is to prove that

$$\sum |u_n|^2(1+|n|^2)^t \leq \sum |u_n|^2(\varepsilon(1+|n|^2)^t + C(\varepsilon)(1+|n|^2)^r)$$

which is true if

$$(1+|n|^2)^s \le \varepsilon(1+|n|^2)^t + C(\varepsilon)(1+|n|^2)^r \text{ for all } n.$$

That is, if

$$1 \le \varepsilon(1+|n|^2)^{t-s} + C(\varepsilon)(1+|n|^2)^{r-s}.$$

Now for $x > 0$, we have the identity

$$x^s \le x^t + x^r \text{ or } 1 \le x^{t-s} + x^{r-s}.$$

Putting $x = \alpha x$, we have $1 \le \alpha^{t-s}x^{t-s} + \alpha^{r-s}x^{r-s}$. Choose α such that $\alpha^{t-s} = \varepsilon$. Hence $\alpha^{r-s} = \varepsilon^{\frac{r-s}{t-s}} = C(\varepsilon)$ (say), so that $1 \le \varepsilon x^{t-s} + C(\varepsilon)x^{r-s}$. Putting $x = 1+|n|^2$ we get

$$1 \le \varepsilon(1+|n|^2)^{t-s} + C(\varepsilon)(1+|n|^2)^{r-s}.$$

□

10 Regularity Theorem in the General Case

Theorem 52. *Let Ω be an open set in $\mathbb{R}^n$ and L an elliptic linear partial differential operator of order k. Let $u \in \mathcal{D}'(\Omega)$ and $Lu = f$ where $f \in C^\infty(\Omega)$. Then $u \in C^\infty(\Omega)$.*

Proof. Let $L = \sum_{|\alpha|\le k} a_\alpha D^\alpha$, $a_\alpha \in C^\infty(\Omega)$. To prove that given $x_0 \in \Omega$, u is C^∞ around x_0. Choose neighbourhoods $V \subset \overline{V} \subset U \subset Q$ of x_0 such that Q is one of the fundamental cubes. In fact we choose r small enough so that $B(x_0, 2r) \subset Q$ and put $V = B(x_0, r)$, $U = B(x_0, 2r)$. Let $g \in C^\infty(\Omega)$ be such that g takes values in $[0,1]$, $g = 1$ inside $B(x_0, r)$ and $g = 0$ outside $B(x_0, 2r)$. Let $L_0 = \sum_{|\alpha|\le k} a_\alpha(x_0)D^\alpha$. Let $\tilde{l} = gL + (1-g)L_0$. Then $\tilde{l} \equiv L$ inside $B(x_0, r)$ and $\tilde{l} \equiv L_0$ outside $B(x_0, 2r)$. Since the coefficients of $\tilde{l}$ are constant on the boundary of Q, $\tilde{l}$ defines a differential operator A on the torus $\mathbb{T}$. Also we can show that if the support of g is sufficiently small, then $\tilde{l}$ is elliptic and hence A is elliptic. Choose a C^∞ function $g_1 : \Omega \to [0,1]$ such that

$$g_1 = \begin{cases} 1 & \text{inside } B(x_0, \frac{r}{2}) \\ 0 & \text{outside } B(x_0, \frac{3r}{4}). \end{cases}$$

Now we observe that inside $B(x_0, \frac{r}{2})$, $g_1 u$ as a distribution on $\mathbb{T}$ is the same as u as a distribution on Ω. That is, if $h \in C_c^\infty(\Omega)$, with support of h a subset of $B(x_0, \frac{r}{2})$ and $\tilde{h} \in C^\infty(\mathbb{T})$ is got by extending h periodically outside Q, then $(g_1 u, \tilde{h}) = (g_1 u, h)$. Now for all $\phi \in C^\infty(\mathbb{T})$,

$$\begin{aligned}(A(g_1u),\phi) &= (g_1u, A^*\phi)\\ &= (g_1u, A^*(g\phi)) + (g_1u, A^*(1-g)\phi)\\ &= (g_1u, A^*(g\phi))\\ &= (g_1u, L^*(g\phi))\\ &= (u, L^*(g_1g\phi)) + (u, P^*\phi)\\ &= (u, L^*(g_1\phi)) + (u, P^*\phi)\\ &= (Lu, g_1\phi) + (u, P^*\phi)\\ &= (f, g_1\phi) + (Pu, \phi)\\ &= (fg_1, \phi) + (Pu, \phi)\\ &= (g_1f + Pu, \phi).\end{aligned}$$

In the above, P^* is a differential operator on Ω of order less than or equal to $k-1$ with coefficients having support inside the support of g.

This is true for all $\phi \in C^\infty(\mathbb{T})$. Therefore

$$A(g_1u) = g_1f + Pu. \tag{12}$$

Now let $u \in H_t$. Then $Pu \in H_{t-k+1}$. Therefore Eq. 12 gives $A(gu) \in H_{t-k+1}$. Hence the elliptic regularity for the torus gives $g_1u \in H_{t-k+1+k} = H_{t+1}$. But $g_1u = u$ inside $B(x_0, \frac{r}{2}) \subset \mathbb{T}$. Therefore $u \in H_{t+1}$ inside $B(x_0, \frac{r}{2})$. Repeating the above process we see that $u \in H_t$ for all t, inside $B(x_0, \frac{r}{2})$. Therefore u and hence g_1u is C^∞ inside $B(x_0, \frac{r}{2}) \subseteq \mathbb{T}$. Therefore u as a distribution on Ω is C^∞ inside $B(x_0, \frac{r}{2})$. That is, for all $\phi \in C_c^\infty(\Omega)$ with support $\phi \subseteq B(x_0, \frac{r}{2})$, $u(\phi) = \int g\phi$ where $g \in C_c^\infty(\Omega)$. Now we use a partition of unity argument to show that u is globally given by a C^∞ function. Cover Ω by balls of radius $\frac{r}{2}$ and choose a partition of unity subordinate to the above covering. Let ϕ be a C^∞ with support inside a compact set K. Then there exists an open set $W \supseteq K$ and $\psi_1, \dots \psi_m$ such that $\sum \psi_1(x) = 1$ for all $x \in W$. Therefore $\phi = \sum_{i=1}^m \phi\psi_i$ and

$$u(\phi) = \sum_{i=1}^m u(\phi\psi_i) \tag{13}$$

Now corresponding to each $\alpha \in \Lambda$ choose a $g_\alpha \in C^\infty(\Omega)$ such that for $\eta \in C_c^\infty(\Omega)$ with support of $\eta \subset$ support ψ_α, $u(\eta) = \int g_\alpha \eta$. Then Eq. 13 shows that u is given by $u(\phi) = \int \phi f$, where $f = \sum_{\alpha\in\Lambda} g_\alpha\psi_\alpha \in C^\infty(\Omega) \cap L^1_{\text{loc}}$. That is, $u \in C^\infty(\Omega)$. □