# Two Interesting Examples in Topology

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## 1 First Example

This is a write-up of an interesting example constructed in MTTS 2012. I have not checked the details and some of the claims may be wrong! Please go through the article and let me know. If you find a better way of doing this, send me your comments.

Let $X = \mathbb{R}$. Let $\mathcal{T}_d$ denote the standard topology on $\mathbb{R}$. Let $\mathcal{S}$ be the collection of all subsets of the form $G := U \setminus A$ where $U \in \mathcal{T}_d$ and $A$ is a countable subset of $\mathbb{R}$. Let $\mathcal{T}$ be the smallest topology containing $\mathcal{S}$. It is easy to see that $\mathcal{T} = \mathcal{S}$. We shall always denote elements of $\mathcal{T}$ as $G = U \setminus A$ etc.

The space $(X, \mathcal{T})$ is Hausdorff, as $\mathcal{T}_d \subset \mathcal{T}$. It is not first countable. For, if $\{G_n : n \in \mathbb{N}\}$ is a local basis at some point $x \in X$, then $G_n = U_n \setminus A_n$. Choose $x_n \in G_n$, $x_n \neq x$. The set $V := G_k \setminus \{x_n : n \in \mathbb{N}\}$ is an open set containing $x$. No $G_n$ can be a subset of $V$.

Another way of seeing this is as follows. Let $b \notin \cup_n A_n$. Then $\mathbb{R} - \{b\}$ is an open set containing $x$. No $G_n$ is contained in this set. Or, let $A := \cup_n A_n$. Let $U \ni x$ be any set in $\mathcal{T}_d$. Then $U \setminus A$ is an open set containing $x$. It does not contain any $G_n$.

Let $(x_n)$ be a sequence in $X$ converging to $x$ in $\mathcal{T}$. Let $G_n := (x - 1/n, x + 1/n) \setminus \{x_n : x_n \neq x, n \in \mathbb{N}\}$. Then $G_n$ does not contain any element of the sequence other than $x$. This shows that any convergent sequence in $(X, \mathcal{T})$ is eventually constant.

Let $x$ be a cluster point of $E \subset X$. Since $\mathcal{T}$ is Hausdorff, there exists a sequence of distinct elements of $E$ converging to $x$. This is impossible, hence $E$ has no cluster point.

> This shows why the concept of cluster point is not the appropriate concept/definition to deal with the closures of subsets and why it is better to work with limit-points, as I define. The importance of the notion of cluster point lies therefore in the Bolzano-Weierstrass property used in compactness, I guess.

An immediate fall-out of this is that any limit point of a countable set lies in it and hence no countable set can be dense. Hence the space $(X, \mathcal{T})$ cannot be separable.

Let $K$ be a compact subset of $(X, \mathcal{T})$. If $K$ is infinite, let $A := \{x_n : n \in \mathbb{N}\}$ be a countably infinite subset of $K$. Then $\{\mathbb{R} \setminus \{x_n\} : n \in \mathbb{N}\}$ is an open cover of $K$ which has no finite subcover. Hence an infinite set cannot be compact. We conclude that any compact subset of $(X, \mathcal{T})$ is finite.

This immediately leads us to suspect that $(X, \mathcal{T})$ is not path-connected. The image $[\gamma]$ of a path $\gamma\colon [0,1] \to (X, \mathcal{T})$ is a compact connected subset of $(X, \mathcal{T})$. Hence $[\gamma]$ is a finite set. Can this be connected unless it is a singleton?

Let $E = \{x_1, \dots, x_n\}$ be a finite set with $n \geq 2$. Let $f\colon E \to \{\pm 1\}$ be the function defined as $f(x_j) = 1$ for $1 \leq j < n$ and $f(x_n) = -1$. Then $f$ is continuous. For, the sets $G_1 := \mathbb{R} \setminus \{x_n\}$ and $G_n := \mathbb{R} \setminus \{x_j : 1 \leq j \leq n-1\}$ will attend to the continuity of $f$. Thus we conclude that $[\gamma]$ is a singleton. Hence the space is not path-connected.

Is it connected? I guess so. It seems that if $G = U \setminus A$, then $x$ is a limit point of $G$ iff it is a limit point of $U$. Hence if $G$ is both open and closed, then so is $U$.

Any open cover of $X$ admits a countable subcover. For, if $\{G_i\}$ is an open cover, then $\{U_i\}$ is an open cover of $(\mathbb{R}, \mathcal{T}_d)$ and hence it admits a countable subcover, say $\{U_n\}$. Then $\{G_n\}$ is an open cover of $X$ but for a countable subset. Hence we can include a countable number of elements from $\{G_i\}$ to cover this countable subset.

Are there any other interesting observations about this space?

## 2 Cluster Point Versus Limit Point

We say that a subset $F \subset \mathbb{N}$ is *small* if $\sum_{k \in F} \frac{1}{k}$ is convergent. The empty set is defined to be small. The following are easy to see.

1. Any finite subset $F \subset \mathbb{N}$ is small.
2. If $S$ is small and $T \subset S$, then $T$ is small.
3. If $F_k$ is small for $1 \leq k \leq N$, then $F := \cup_{k=1}^N F_k$ is small.
4. If $S$ is an infinite subset of $\mathbb{N}$, there exists $T \subset S$ such that $T$ is an *infinite* small set.
   To see this, observe that for each $k \in \mathbb{N}$ there exists $n_k \in S$ such that $n_k > 2^k$.
5. $\mathbb{N}$ is not small.

Let $X := \mathbb{N} \cup \{0\}$. We declare a subset $U \subset X$ open if (i) either $U \subset \mathbb{N}$ or (ii) if $0 \in U$, then $N \setminus U$ is small. It is easy to see that the collection of open sets defines a Hausdorff topology on $X$. While the subspace topology on $\mathbb{N}$ is discrete, the topology on $X$ is not.

We claim that any compact subset $K \subset X$ is finite. For, if $K$ is infinite and is a subset of $\mathbb{N}$, then $\{\{x\} : x \in K\}$ is an infinite open cover of $K$ which does not admit a finite subcover. If $0 \in K$ and $K$ is infinite, we write $K = \{0\} \cup L$, $L \subset \mathbb{N}$. Note that $L$ is an infinite set.

Case 1. $L$ is infinite and small. Then $X \setminus L \cup \{\{l\} : l \in L\}$ is an open cover of $K$ which has no finite subcover.

Case 2. $L$ is infinite and not small. Then by Item 4, there exists an infinite subset, say $T \subset L$ which is small. Note that $L \setminus T$ is infinite. The collection $X \setminus T$ along with $\{\{x\} : x \in L \setminus T\}$ is an open cover of $K$ which admits no finite subcover.

It follows therefore that any convergent sequence in $X$ is eventually constant. The point 0 is a cluster point of the set $\mathbb{N}$. However there is no sequence in $\mathbb{N}$ that converges to 0.

In the terminology of my book, "Topology of Metric Spaces", 0 is a cluster point of $\mathbb{N}$ and also a limit point of $N$. However there is no sequence in $\mathbb{N}$ that converges to 0. Note that this cannot happen in a metric space.

**Remark 1.** This example also brings out the inadequacy of sequences in questions of general topological spaces.