The Exponential Map and Riemann Surface of the Logarithm

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We let $\mathbb{C}^*$ denote the set of nonzero complex numbers.

Definition 1. Let $z \in \mathbb{C}^*$. We define $\operatorname{Arg}(z)$ to be the set $\{t \in \mathbb{R} : z = |z| \exp(it)\}$. Any $t \in \operatorname{Arg}(z)$ is called an argument of z.

The following result is well-known. It is proved using some of fundamental facts about the exponential function, the trigonometric function cos and sin and their periodicity. Any good book on complex analysis should have a proof of this. So, we omit the proof.

Proposition 2. *Let $z \in \mathbb{C}$ be nonzero. Then there exists a unique $\theta \in [0, 2\pi)$ such that $z = |z| \exp(i\theta)$. Also, $\operatorname{Arg}(z) = \{\theta + n2\pi : n \in \mathbb{Z}\}$.* □

The following lists some of the basic facts about the arguments of nonzero complex numbers.

Ex. 3. Let $z, w \in \mathbb{C}^*$. Let $\theta \in \operatorname{Arg}(z)$ and $\varphi \in \operatorname{Arg}(w)$. Then
(1) $-\theta \in \operatorname{Arg}(z^{-1})$.
(2) $\theta + \varphi \in \operatorname{Arg}(zw)$.
(3) $\theta - \varphi \in \operatorname{Arg}(z/w)$.

Given an open set $U \subset \mathbb{C}$, we say that there exists a continuous argument on U if there exists a continuous function $\theta \colon U \to \mathbb{C}$ such that $z = |z|e^{i\theta(z)}$ for all $z \in U$. The following lemma says that there is no continuous argument on $\mathbb{C}^*$.

Lemma 4. *There exists no continuous argument on $\mathbb{C}^*$.*

Proof. Assuming such a θ exists, consider $f \colon [0, 2\pi] \to \mathbb{R}$ by setting

$$f(t) := [\theta(e^{it}) + \theta(e^{-it})]/2\pi.$$

Then f is a real valued continuous function on $[0, 2\pi]$. Since $2\pi f(t)$ is a choice of $\operatorname{Arg}(e^{it}e^{-it})$ and hence of $\operatorname{Arg}(1)$, it is integer valued continuous function on the interval $[0, 2\pi]$. By intermediate value theorem, it is a constant. In particular, $f(0) = f(\pi)$. This implies that $[\theta(1) + \theta(1)]/2\pi = [\theta(-1) + \theta(-1)]/2\pi$. This implies that $\theta(1) = \theta(-1)$, which is impossible. For, $\theta(1) \in \operatorname{Arg}(1)$, which is the set of even integral multiples of π while $\theta(-1) \in \operatorname{Arg}(-1)$ which is the set of odd integral multiples of π. □

However, if we are ready to ignore a closed half-line starting from 0, we can get a continuous choice θ of Arg.

Let $\alpha \in \mathbb{R}$ and let $L_\alpha := \{te^{i\alpha} : t \geq 0\}$. Note that a nonzero $z \in \mathbb{C} \setminus L_\alpha$ iff $\alpha \notin \text{Arg}\,(z)$. For $z \notin L_\alpha$, there is a unique choice of $\text{Arg}\,(z)$ in the interval $(\alpha, \alpha + 2\pi)$. We denote this choice by $\arg_\alpha \colon \mathbb{C} \setminus L_\alpha \to (\alpha, \alpha + 2\pi)$.

Theorem 5. $\arg_\alpha$ *is continuous on* $\mathbb{C} \setminus L_\alpha$.

Proof. Let $\theta := \arg_0$. Then $\mathbb{C} \setminus L_0 = H_0 \cup H_1 \cup H_2$ where $H_0 := \{x + iy : y > 0\}$, $H_1 := \{x + iy : x < 0\}$ and $H_2 := \{x + iy : y < 0\}$. We show that θ is continuous on each of the open half-planes H_i, $0 \leq i \leq 2$.

If $z \in H_0$, then $\text{Im}\,(z) = |z| \sin(\theta(z))$. So, $\sin(\theta(z)) > 0$. Since θ takes values only in $(0, 2\pi)$, this means that $\theta(z) \in (0, \pi)$ for such z. Now, $\cos \colon (0, \pi) \to (-1, 1)$ is strictly decreasing. Hence it has a continuous inverse $\cos^{-1} \colon (-1, 1) \to (0, \pi)$. Hence

$$\theta(z) = \cos^{-1}(\cos(\theta(z))) = \cos^{-1}(\text{Re}\,(z)/|z|), \qquad z \in H_0.$$

Since the RHS is continuous, it follows that θ is continuous on H_0.

Similar reasoning leads us to the following expressions for θ restricted to H_1 and H_2:

$$\begin{aligned} \theta(z) &= \pi - \sin^{-1}\left(\frac{\text{Im}\, z}{|z|}\right), \qquad z \in H_1 \\ \theta(z) &= 2\pi - \cos^{-1}\left(\frac{\text{Re}\, z}{|z|}\right) \qquad z \in H_2. \end{aligned}$$

The continuity of θ on H_1 and H_2 follows from these expressions as earlier. Thus the restrictions of θ to the open sets H_0, H_1, H_2 are continuous. Since the union of these open sets is $\mathbb{C} \setminus L_0$, an appeal to the gluing lemma establishes the continuity of θ.

General case: Consider the map $f(z) := z \exp(-i\alpha)$ which maps $\mathbb{C} \setminus L_\alpha$ onto $\mathbb{C} \setminus L_0$. Then one shows that $\arg_\alpha(z) = \alpha + \theta(f(z))$. □

If $x \in \mathbb{R}$ is positive, we know that there exists a unique $y \in \mathbb{R}$ such that $x = \exp(y)$. We let $y := \log(x)$. Thus, we have a function $\log \colon \mathbb{R}_+ \to \mathbb{R}$ defined by $\exp(\log(x)) = x$ for all $x \in \mathbb{R}_+$. Since exp is a (continuous) increasing function, log is continuous (and increasing).

Definition 6. Given $z \in \mathbb{C}^*$, any complex number w such that $\exp(w) = z$ is called a *logarithm* of z. We let $\text{Log}\,(z)$ stand for the set of all logarithms of a nonzero $z \in \mathbb{C}$. This set is nonempty.

Example 7. If $z \in \mathbb{C}^*$, then $w := \log(|z|) + it$ where $t \in \text{Arg}\,(z)$ is a logarithm of z. Furthermore, any logarithm of z is of this form.

We say that there is a continuous logarithm on an open set $U \subset \mathbb{C}^*$ if there is a continuous function $F \colon U \to \mathbb{C}$ such that $z = \exp(F(z))$ for all $z \in U$.

Ex. 8. Let $U \subset \mathbb{C}^*$ be open. Then there is a continuous argument on U iff there is a continuous logarithm on U. Hence conclude that there is no continuous logarithm on $\mathbb{C}^*$.

We now characterize the open sets of $U \subset \mathbb{C}^*$ which admit continuous logarithm using a topological concept.

Theorem 9. *The exponential map* $\exp\colon \mathbb{C} \to \mathbb{C}$ *maps the open strip*

$$S_{\alpha,n} := \{z \in \mathbb{C} : \alpha + 2\pi n < \operatorname{Im} z < \alpha + 2\pi(n+1)\}$$

homeomorphically onto $\mathbb{C} \setminus L_\alpha$ *for each* $\alpha \in \mathbb{R}$.

Proof. The map $\exp\colon S_{\alpha,n} \to \mathbb{C} \setminus L_\alpha$ is a continuous bijection. Its inverse is given by $F(z) = \log|z| + 2n\pi + \arg_\alpha(z)$. Here $\log|z|$ stands for the real logarithm of the positive number $|z|$. Hence the map $z \mapsto \log|z|$, being the composite of two continuous functions, is continuous. Since $\arg_\alpha$ is continuous by the last theorem, the continuity of F follows. □

Definition 10. Let $f\colon X \to Y$ be a continuous onto map. An open set $V \subset Y$ is said to be *evenly covered* by f if the inverse image $f^{-1}(V)$ is a disjoint union $\cup_{i \in I} U_i$ of open sets in X each of which is mapped homeomorphically onto V by f.

We say that f is *covering map* if each $y \in Y$ has an open neighbourhood which is evenly covered by f.

Proposition 11. *Let* $U \subset \mathbb{C}^*$ *be such that* $U \cap L_\alpha = \emptyset$ *for some* $\alpha \in \mathbb{R}$. *Then* U *is evenly covered by the exponential map.*

Proof. For each integer $n \in \mathbb{Z}$, we define

$$V_n := \{z \in \mathbb{C} : \alpha + 2\pi n < \operatorname{Im} z < \alpha + 2\pi(n+1)\} \cap \exp^{-1}(U).$$

Then each V_n is open and $\exp^{-1}(U)$ is the union of V_n. By the last theorem, the restriction of the exponential map to the strip $S_{\alpha,n}$ sets up a bijection between the open subsets of the strip with those of $\mathbb{C} \setminus L_\alpha$. In particular, the open set V_n is mapped homeomorphically onto U for each n. Thus U is evenly covered by the exponential map. □

Corollary 12. *The open set* $\mathbb{C}^*$ *is not evenly covered by the exponential map.*

Proof. If $\mathbb{C}^*$ is evenly covered by the exponential map, then its inverse image under exp is $\mathbb{C}$. Hence $\mathbb{C}$ is the disjoint union of open sets, each of which will be homeomorphic to $\mathbb{C}^*$. This contradicts the fact that $\mathbb{C}$ is connected. □

Lemma 13. *Let* $U \subset \mathbb{C}^*$ *be open. Let* $F\colon U \to \mathbb{C}$ *be a continuous logarithm on* U. *Then* $F(U)$ *is open in* $\mathbb{C}$.

Proof. This follows easily if we recall some well-known results from complex analysis. F, being the continuous inverse of the holomorphic function exp, is itself holomorphic. Also, F cannot be a constant on any of the connected components of U. (Why?) Now we can appeal to the open mapping theorem to derive the result. However, we indicate an elementary along the lines developed so far.

Let $S_\alpha := \{z \in \mathbb{C} : \alpha < \operatorname{Im} z < \alpha + 2\pi\}$ for some $\alpha \in \mathbb{R}$. Now,

$$F^{-1}(S_\alpha) = \{z \in U : F(z) \in S_\alpha\} = \exp(S_\alpha \cap F(U)).$$

This shows that $\exp(S_\alpha \cap F(U))$ is an open subset of U and hence of $\mathbb{C}$. Clearly, $\exp(S_\alpha \cap F(U)) \subset \mathbb{C} \setminus L_\alpha$. Since exp is a homeomorphism of S_α onto $\mathbb{C} \setminus L_\alpha$, it follows that $S_\alpha \cap F(U)$ is open in $\mathbb{C}$. If we observe that $F(U)$ is the union of sets of the form $S_\alpha \cap F(U)$, we conclude that $F(U)$ is open. □

Theorem 14. *An open set $U \subset \mathbb{C}^*$ is evenly covered by the exponential map iff there is a continuous logarithm on U.*

Proof. Assume that U is evenly covered by exp. Let $\exp^{-1}(U) = \cup_i V_i$. Fix an i and let F be the inverse of the homeomorphism $\exp\colon V_i \to U$. Then F is as required.

Conversely, let F as in the statement exist. Then by the last lemma, $F(U)$ is open in $\mathbb{C}$. Call this open set V_0. We let $V_n := V_0 + i2\pi n$, the translate of V_0 by $i2\pi n$. Then V_n's are open, they are pairwise disjoint and their union is $\exp^{-1}(U)$. The restriction of exp to any V_n maps V_n homeomorphically onto U. Hence U is evenly covered. □

The inverse maps $F_n\colon U \to V_n$ are known as the branches of the logarithm on U.

The following gives a more sophisticated solution to Exercise 8.

Corollary 15. *There exists no continuous logarithm on $\mathbb{C}^*$.*

Proof. If there exists a continuous logarithm on $\mathbb{C}^*$, then $\mathbb{C}^*$ is evenly covered by the exponential map, by the last theorem. This contradicts Corollary 12. □

We are now ready to construct the Riemann surface for the logarithm function. First of all, a definition.

Definition 16. A topological space X is said to be a Riemann surface if there exists an open cover $\{U_i : i \in I\}$ of X, open subset $V_i \subset \mathbb{C}$ for each $i \in I$ and a homoemorphism $\varphi_i\colon V_i \to U_i$ with the following property: whenever $U_i \cap U_j \neq \emptyset$, the map $\varphi_j^{-1} \circ \varphi_i\colon \varphi_i^{-1}(U_i \cap U_j) \to \varphi_j^{-1}(U_i \cap U_j)$ is holomorphic. (Note that the domain and codomain are open subsets of $\mathbb{C}$, since φ_i's are homeomorphism.) The maps φ_i are called the parametrizing maps. We say that U_i is parametrized by φ_i. The maps $\varphi_j^{-1} \circ \varphi_i$ are called the transition maps.

Example 17. Any open subset $U \subset \mathbb{C}$ is a Riemann surface.

Example 18. A most important example, which is encountered, albeit in disguise, in any first course in complex analysis is the extended complex plane $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. The topology on $\mathbb{C}_\infty$ is clear once we recognize it as the one point compactification of $\mathbb{C}$. We take $U_0 = \mathbb{C}$ and $U_\infty = \mathbb{C}_\infty \setminus \{0\}$. The parametrizing maps $\varphi_0\colon \mathbb{C} \to U_0$ is the identity map while $\varphi_1\colon \mathbb{C} \to U_1$ is the map $\varphi(z) = \begin{cases} 1/z & \text{if } z \neq 0 \\ \infty & \text{if } z = 0. \end{cases}$ One shows that $\mathbb{C}_\infty$ is a Riemann surface. (The transition map here is $z \mapsto 1/z$ from $\mathbb{C}^*$ to itself.)

On any Riemann surface X, we can talk of a function $f\colon X \to \mathbb{C}$ being analytic or holomorphic. It is analytic if $f \circ \varphi_x\colon V_x \to \mathbb{C}$ is so for all $x \in X$. We say that $p \in X$ is a zero of f of order k if the function $f \circ \varphi_p\colon V_p \to \mathbb{C}$ has a zero at $\varphi_p(p)$ of order k. Similarly other notions can be defined. Now, a perceptive reader may recognize that this is precisely what we have been doing all along when we dealt with the behaviour of functions at point at infinity.

We now construct the Riemann surface of log as follows. Let

$$S := \{(z, w) \in \mathbb{C}^2 : z \in \mathbb{C}^* \text{ and } \exp(w) = z\}.$$

Thus S is the set of all ordered pairs of points such that the second element is a logarithm of the first. A better way of recognizing S is to think of it as the graph of exp written in reverse ordered pairs: $(\exp(w), w)$. We define π_1, π_2 be the natural projection maps. Note that the map $w \mapsto (\exp(w), w)$ to S is a homeomorphism.

We show that given $(z_0, w_0) \in S$, there exists a parametrized neighbourhood of (z_0, w_0). Let U be an evenly covered open neighbourhood of z_0 in $\mathbb{C}^*$. (Why do they exist?) We can therefore choose a branch of continuous logarithm F on U in such a way that $F(z_0) = w_0$. Define $\varphi\colon U \to S$ by $\varphi(z) = (z, F(z))$. This gives a parametrization of the surface S around (z_0, w_0). The sheets of the Riemann surface S that cover U are the open sets V_n where

$$V_n = \{(z, w) \in S : z \in U \text{ and } w \in F(U) + i2\pi n\}.$$

The function π_2 is the 'logarithm' defined globally on S and is holomorphic on S: For, let $(z_0, w_0) \in S$. Let $\varphi\colon V \to U$ be a parametrization of an open set containing (z_0, w_0). Then the composite $\pi_2 \circ \varphi\colon V \to \mathbb{C}$ is $\pi_2 \circ \varphi(z) = \pi_2(z, F(z)) = F(z)$, a logarithm of z. It is clearly holomorphic. It is also a homeomorphism of S onto $\mathbb{C}$. Thus while there exists no holomorphic logarithm on $\mathbb{C}^*$, we have produced a holomorphic logarithm on S which is homeomorphic to $\mathbb{C}$!

We now show what happens to a loop in $\mathbb{C}^*$ winding around 0 when lifted to S. Consider the loop $\gamma\colon [0, 1] \to \mathbb{C}^*$ defined by $\gamma(t) = \exp(2\pi i t)$. Let us consider the lift through $(1, 0)$ of $1 = \gamma(0) = \gamma(1)$ in S. We wish to lift γ as a curve starting from $(1, 0)$. Call the lift $\tilde{\gamma}$. It is easily seen that the lift is given by $\tilde{\gamma}(t) = (\exp(2\pi i t), 2\pi i t)$. In particular, $\tilde{\gamma}(1) = (1, 2\pi i)$. Thus the curve $\tilde{\gamma}$ starting at one sheet ended in a different sheet.