Approximate Identities

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Definition 1. A sequence (K_n) of functions $K_n\colon \mathbb{R} \to \mathbb{R}$ is called an *approximate identity* if it has the following properties:

1. $K_n(x) \geq 0$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.
2. K_n are continuous.
3. $\int_{-\infty}^{\infty} K_n(x)\, dx = 1$ for all n.
4. Given $\varepsilon > 0$ and $\delta > 0$ there exists $N \in \mathbb{N}$ such that

$$\int_{-\infty}^{-\delta} K_n(x)\, dx + \int_{\delta}^{\infty} K_n(x)\, dx < \varepsilon, \text{ for } n \geq N.$$

Property 3) means that the area under the graph of K_n is 1. Property 4) means that the area is concentrated around $x = 0$ for n sufficiently large.

Ex. 2. Let $K\colon \mathbb{R} \to \mathbb{R}$ be continuous. Assume that $K(x) = 0$ for $|x| \geq R$ for some R and that $\int_{-\infty}^{\infty} K(x)\, dx = 1$. Define $K_n(x) = nK(nx)$ for $x \in \mathbb{R}$. Then (K_n) is an approximate identity. (This is a very useful and most important way of generating approximate identities.)

Definition 3. Let $f, g\colon \mathbb{R}^n \to \mathbb{R}$ be functions. Assume that the integral $\int_{\mathbb{R}^n} f(x-y)g(y)\, dy$ makes sense (either as a Lebesgue integral or as a Riemann integral) for all $x \in \mathbb{R}^n$. Then we denote it by $f * g(x)$. The function $f * g$ so obtained is called the *convolution* of f and g.

Ex. 4. Let $f := \chi_{[0,1]}$ be the characteristic function of the interval $[0, 1]$. Compute $f * f$.

Theorem 5. *Let (K_n) be an approximate identity. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Assume that $f(x) = 0$ for $|x| \geq R$ for some R. Let (f_n) be defined as follows:*

$$f_n(x) := K_n * f(x) := \int_{-\infty}^{\infty} K_n(x-y)f(y)\, dy, \qquad x \in \mathbb{R}.$$

Then f_n converges uniformly to f on $\mathbb{R}$.

Proof. By the change of variable $z := x - y$ we have

$$f_n(x) = \int_{-\infty}^{\infty} K_n(z)f(x-z)\, dz.$$

Also observe that $f(x) = \int_{-\infty}^{\infty} f(x)K_n(y)\, dy$. Hence we have

$$f_n(x) - f(x) = \int_{-\infty}^{\infty} K_n(y)\left[f(x-y) - f(x)\right]\, dz. \tag{1}$$

It is easily seen that f is uniformly continuous on $\mathbb{R}$. Let $\varepsilon > 0$ be given. For this ε, by the uniform continuity of f there is a δ such that $|f(x+h) - f(x)| < \varepsilon$ for $|h| < \delta$. We split the domain of integral on the RHS into three intervals: $\mathbb{R} = (-\infty, -\delta] \cup [-\delta, \delta] \cup [\delta.\infty)$. Thus we have

$$f_n(x) - f(x) = \left(\int_{-\infty}^{-\delta} + \int_{-\delta}^{\delta} + \int_{\delta}^{\infty} \right) [K_n(y)\,(f(x-y) - f(x))]\, dy. \tag{2}$$

For ε and δ as above, we choose N given by property 4) of (K_n). The first and the third integrals in Eq. 2 are estimated using the upper bound M for f and the 4th property of (K_n):

$$\begin{aligned} |f_n(x) - f(x)| &\le \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \right) [K_n(y)\,(f(x-y) - f(x))]\, dy \\ &\le 2M\varepsilon. \end{aligned} \tag{3}$$

$$\tag{4}$$

The second one is estimated by uniform continuity and the 3rd property of (K_n):

$$\begin{aligned} |f_n(x) - f(x)| &\le \int_{-\delta}^{\delta} K_n(y) |f(x-y) - f(x)|\, dy \\ &\le \varepsilon \int_{-\infty}^{\infty} K_n(y)\, dy = \varepsilon. \end{aligned} \tag{5}$$

$$\tag{6}$$

From Eq. 3 and Eq. 5 we conclude that $|f_n(x) - f(x)| \le (2M+1)\varepsilon$. □

Ex. 6. Let φ be C^∞-function which is zero outside a bounded interval, say, $[-R, R]$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Show that $K * f$ is C^∞. *Hint:* Observe that $(K * f)' = K' * f$.

Ex. 7. Let f be as in the last exercise. Show that there exists a sequence (f_n) of C^∞ functions which converge uniformly on compact subsets of $\mathbb{R}$ to f.

Ex. 8. Let Ω be a nonempty open subset of $\mathbb{R}^n$. Show that the set $C_c^\infty(\Omega)$ of infinitely differentiable functions with compact support is dense in $L^p(\Omega)$. *Hint:* The set $C_c(\Omega)$ of continuous functions with compact support is dense in $L^p(\Omega)$

Lemma 9. *Let $f\colon [0,1] \to \mathbb{R}$ be continuous and $f(0) = 0 = f(1)$. Given $\varepsilon > 0$ there exists a polynomial p such that $|f(x) - p(x)| < \varepsilon$ for $x \in [0,1]$.*

Proof. Define (k_n) as follows: $k_n(x) := \begin{cases} (1-x^2)^n & |x| \le 1, \\ 0 & |x| > 1 \end{cases}$ Then k_n's are continuous, $k_n \ge 0$ and even. Also, $\int_{\mathbb{R}} k_n = \int_{-1}^{1} k_n = C_n < \infty$. If we set $K_n := k_n/C_n$ then (K_n) is an approximate identity. To verify the 4th property, observe that

$$\begin{aligned} C_n &= \int_{-1}^{1} (1-x^2)^n\, dx = 2\int_0^1 (1-x^2)^n\, dx \\ &= 2\int_0^1 (1-x)^n (1+x)^n\, dx \\ &\ge 2\int_0^1 (1-x)^n\, dx = 2/(n+1). \end{aligned}$$

Hence for any $\delta > 0$,

$$\int_\delta^1 K_n(x)\,dx \le \int_\delta^1 (\frac{n+1}{2})(1-\delta^2)^n\,dx \le (\frac{n+1}{2})(1-\delta^2)^n(1-\delta).$$

If we let $r := (1-\delta^2)$ then $0 < r < 1$ so that $\lim_{n\to\infty} nr^n = 0$. □

(K_n) are called the Landau kernels.

Theorem 10. *Let $f\colon [0,1] \to \mathbb{R}$ be continuous. Given $\varepsilon > 0$ there exists a polynomial p such that $|f(x) - p(x)| < \varepsilon$ for $x \in [0,1]$.*

Proof. Consider $g(x) := f(x) - f(0) - x\,(f(1) - f(0))$. Then Lemma 9 can be applied to g. □

Theorem 11 (Weierstrass Approximation Theorem)**.** *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Given $\varepsilon > 0$ there exists a polynomial p such that $|f(x) - p(x)| < \varepsilon$ for $x \in [a,b]$.* □

Ex. 12. Let $f\colon [0,1] \to \mathbb{R}$ be C^1. Show that there exists a sequence of polynomials (p_n) such that $p_n \to f$ in C^1-norm:

$$\sup_{x\in[0,1]} \{|p_n(x) - f(x)|\} + \sup_{x\in[0,1]} \{|p_n'(x) - f'(x)|\} \to 0 \text{ as } n \to \infty.$$

Fejer Kernels

Let $D_n(x) := \sum_{k=-n}^n e^{ikx}$ and $K_n(x) := \dfrac{1}{n+1}\displaystyle\sum_{k=0}^n D_k(x)$. (K_n) is called the *Fejer kernels.*

Ex. 13. Show that

$$K_n(x) = \frac{1}{2(n+1)} \cdot \frac{\sin^2(\frac{(n+1)x}{2})}{\sin^2\frac{x}{2}}.$$

Hint:

$$\sum_{k=0}^n D_k(x) = \sum_{k=0}^n \frac{\sin(k+\frac{1}{2})x}{2\sin\frac{x}{2}} = \frac{1}{2\sin x/2}\Im(\sum_{k=0}^n e^{i(k+1/2)x}) = \frac{1}{2\sin x/2}\Im(e^{ix/2}\frac{1-e^{i(n+1)x}}{1-e^{ix}}).$$

Ex. 14. The sequence $\{K_n\}$ of Fejer kernels has the following properties:

(i) $K_n \ge 0$.

(ii) $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} K_n = 1$.

(iii) Given $\varepsilon > 0$ and $\delta > 0$, there exists N such that if $n \ge N$ then

$$(\int_{-\pi}^{-\delta} + \int_\delta^\pi) K_n < \varepsilon.$$

Hint: To prove (iii) observe that

$$\frac{1}{n}\int_\delta^\pi \frac{\sin^2 nt/2}{\sin^2 t/2}dt \le \frac{1}{n}\int_\delta^\pi \frac{1}{\sin^2 t/2}dt$$

and the last integral is a real number.

Definition 15. A sequence $\{K_n\}$ of real valued continuous functions in $[-\pi, \pi]$ (with period 2π) is called an *approximate identity* **on** $[-\pi, \pi]$ if it has the three properties listed in the last exercise. Thus the sequence $\{K_n\}$ of Fejer kernels is an approximate identity on $[-\pi, \pi]$.

Ex. 16. Let $\{K_n\}$ be an approximate identity on $[-\pi, \pi]$. Let f be a continuous function as $[-\pi, \pi]$ of period 2π. Then $f_n(x) := f * K_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) K_n(x-t) dt$ converges uniformly to f on $[-\pi, \pi]$. *Hint:* Proceed as in the proof of Thm. 5 exploiting the periodicity of the kernels and that of the function f in a change of variable.

Remark 17. Fejer kernels make their appearance in the study of convergence of Fourier series. If we take K_n to be the Fejer kernels in the last exercise then it is known as the Fejer's theorem. It says that the $(C, 1)$-sums of the Fourier series of a continuous periodic functions converges uniformly to the function.

Dirichlet Problem on the Upper Half Plane

Ex. 18. Let $K(x) := \frac{1}{\pi} \frac{1}{1+x^2}$ for $x \in \mathbb{R}$. Then K is continuous, nonnegative, even and $\int_{\mathbb{R}} K = 1$. We let

$$K_y(x) := \frac{1}{y} K\left(\frac{x}{y}\right) = \frac{1}{\pi} \frac{y}{y^2 + x^2}, \qquad y > 0.$$

$(K_y)_{y>0}$ is called the Poisson kernel for the upper half plane. Then $f(x, y) := f_y(x) := K_y * f(x) \to f(x)$ as $y \to 0$ for any continuous integrable function f on $\mathbb{R}$

Ex. 19. Let the notation be as above. Let

$$u(x, y) := K_y * f(x) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{y^2 + (x-y)^2} f(y)\, dy.$$

Then u satisfies the Laplace equation on the upper half plane $\mathbb{H} := \{(x, y) : y > 0\} \subset \mathbb{R}^2$ and is a solution of the boundary value problem

$$\Delta u = 0 \text{ on } \mathbb{H} \text{ and } u(x, 0) = f(x) \text{ for } x \in \mathbb{R}.$$

Hint: Observe that $\frac{y}{y^2+x^2}$ is the imaginary part of $\frac{-1}{z}$, a holomorphic function on $\mathbb{H}$ and hence is harmonic there.

Dirichlet Problem on the Unit Disk

Definition 20. Let the Poisson kernels $(P_r(\theta))$ be defined as follows:

$$\begin{aligned} P_r(e^{i\theta}) &= \frac{1}{2\pi}\sum_{-\infty}^{\infty} r^{|n|}e^{in\theta} \\ &= \frac{1}{2\pi}\frac{1-r^2}{1-2rcos(\theta)+r^2} \\ &= \frac{1}{2\pi}\frac{1-r^2}{|1-re^{i(\theta)}|^2} \\ &= \frac{1}{2\pi}\Re(\frac{1+re^{it}}{1-re^{it}}). \end{aligned}$$

Ex. 21. Show that (P_r) is an approximate identity where the 4th property is rephrased as $r \to 1$ in place $n \to \infty$.

Definition 22. Let $B(0,1)$ be the unit disk in $\mathbb{R}^2$. Let S^1 denote its boundary. Let Δ denote the Laplacian, the second order differential operator defined on any C^2-function u as follows: $\Delta u(x,y) := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. Let f be a continuous function on S^1. The Dirichlet problem is to solve the boundary value problem: Find a function $u \in C(B[0,1])$ such that

$$\Delta u = 0 \text{ on } B(0,1) \text{ and } u(x,y) = f(x,y) \text{ for } (x,y) \in S^1.$$

Ex. 23. Show that the Dirichlet problem on the unit disk in $\mathbb{R}^2$ has a solution if the boundary data f is continuous.

Gaussian Kernels

Ex. 24. Let $K(x) := \frac{1}{\sqrt{\pi}}e^{-x^2}$ for $x \in \mathbb{R}^2$. Then K satisfies the conditions of Exer. 2. Let f integrable and continuous on $\mathbb{R}$. Define

$$u(x,t) := K_{\sqrt{t}} * f(x) := \frac{1}{\sqrt{\pi t}}\int_{\mathbb{R}} f(u)e^{\frac{-(x-u)^2}{t}}\,du.$$

Then u satisfies the initial value problem for the heat equation

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = f(x), \quad x \in \mathbb{R}.$$

Ex. 25. Deduce Weierstrass approximation theorem from the above exercise Ex. 24. (This is Weierstrass original proof.) *Hint:* Note that $u(x,t)$ can be considered as a function of the complex variable z (replacing x by z) and it is entire on $\mathbb{C}$. Now partial sums of the power expansion of this entire functions may do the job.

1 Graphs of Various Approximate Identities

Heat or Gaussian Kernel

$$K_t(x) := \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$$

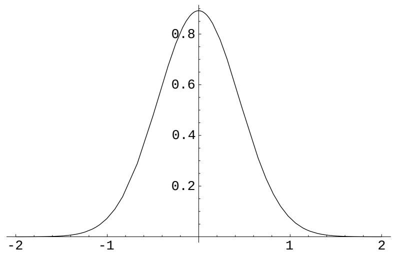

Figure 1: Graph of $K_t(x)$, for $t = 0.1$

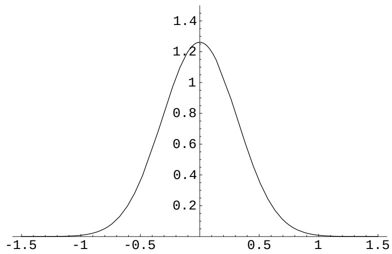

Figure 2: Graph of $K_t(x)$, for $t = 0.05$

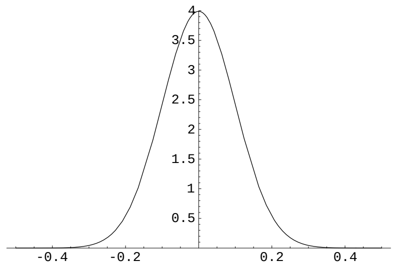

Figure 3: Graph of $K_t(x)$, for $t = 0.005$

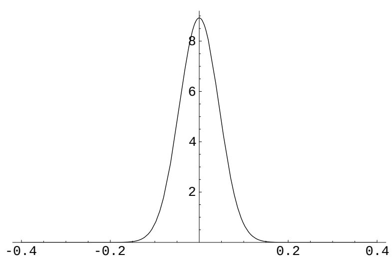

Figure 4: Graph of $K_t(x)$, for $t = 0.001$

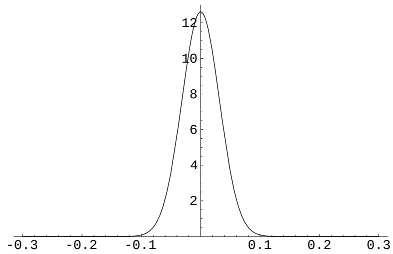

Figure 5: Graph of $K_t(x)$, for $t = 0.0005$

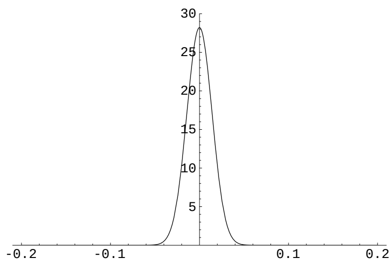

Figure 6: Graph of $K_t(x)$, for $t = 0.0001$

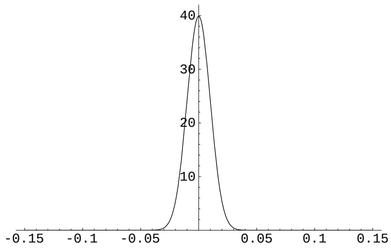

Figure 7: Graph of $K_t(x)$, for $t = 0.00005$

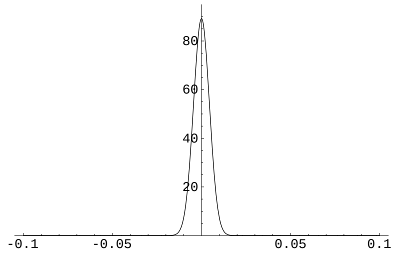

Figure 8: Graph of $K_t(x)$, for $t = 0.00001$

Fejer Kernels

$$K_n(x) = \frac{1}{2(n+1)} \cdot \frac{\sin^2(\frac{(n+1)x}{2})}{\sin^2 \frac{x}{2}}.$$

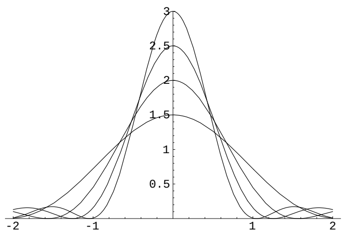

Figure 1: Graph of $K_n(x), n = 2, \cdots, 5$

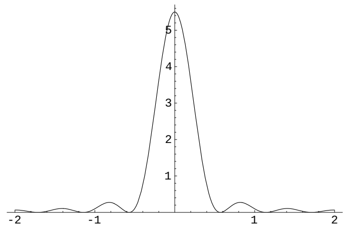

Figure 2: Graph of $K_{10}(x)$

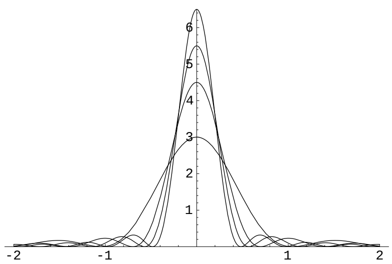

Figure 3: Graph of $K_n(x), n = 5, 8, 10, 12$

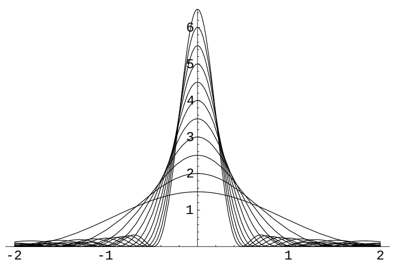

Figure 4: Graph of $K_n(x), n = 2, \cdots, 12$

Poisson Kernels $P_r(t) := \frac{1-r^2}{2\pi}\frac{1}{1-2r\cos t+r^2}$.

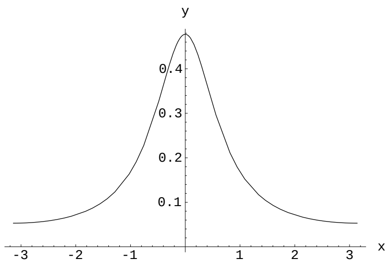

Figure 1: Graph of $P_r(t)$, for $r = 0.5$

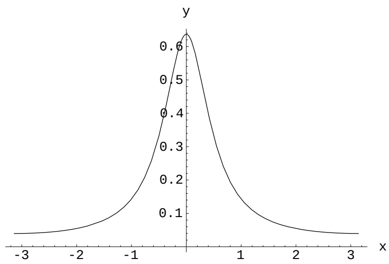

Figure 2: Graph of $P_r(t)$, for $r = 0.6$

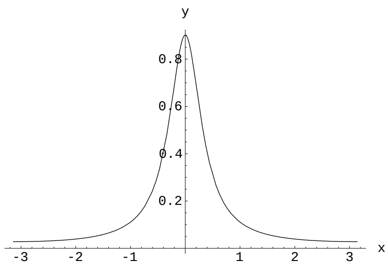

Figure 3: Graph of $P_r(t)$, for $r = 0.7$

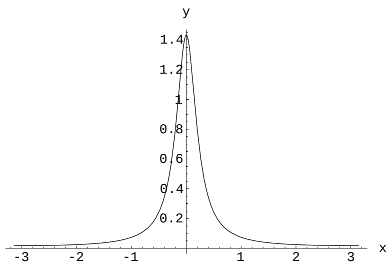

Figure 4: Graph of $P_r(t)$, for $r = 0.8$

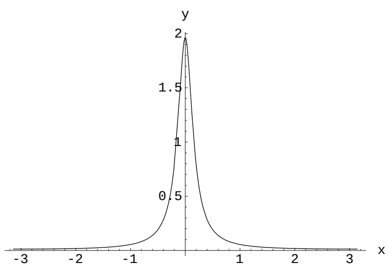

Figure 5: Graph of $P_r(t)$, for $r = 0.85$

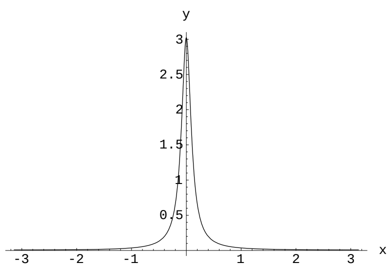

Figure 6: Graph of $P_r(t)$, for $r = 0.9$

Landau Kernels: $L_n(x) := (1-x^2)^n / (\int_{-1}^{1} (1-x^2)^n \, dx), \quad |x| \leq 1$

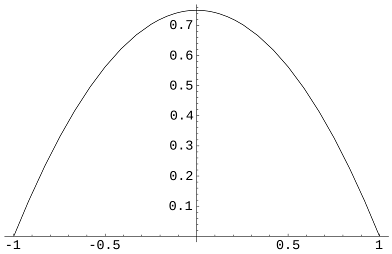

Figure 7: Graph of $L_1(x)$

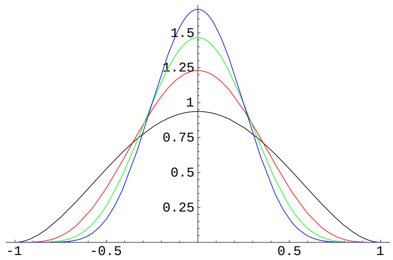

Figure 8: Graph of $L_n(x)$, for $n = 2, 4, 6, 8$

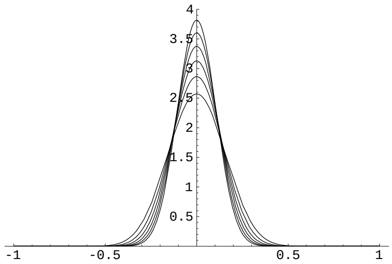

Figure 9: Graph of $L_n(x)$ for $n = 20, 25, 30, 35, 40$