# Finite Sets

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For $n \in \mathbb{N}$, let $I_n := \{1, 2, \dots, n\}$ be the initial segment.

**Definition 1.** A set $A$ is said to be finite if either $A = \emptyset$ or there exists a bijection $f : A \to \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$.

**Lemma 2.** *If $m < n$, there is no one-to-one map of $I_n$ into $I_m$.*

*Proof.* Let $m = 1$ and $n > 1$. No map $f : I_n \to I_1 = \{1\}$ can be 1-1. For, $f(1) = f(n) = 1$ and $n \neq 1$. Thus the result is true for $m = 1$.

Let $P(m)$ be the statements: *Given $n > m$, no map $f : I_n \to I_m$ can be 1-1.*

Thus we have seen P(1) is true. Assume the result $P(m)$. Consider $m+1$. Let $n > m+1$. Let $f : I_n \to I_{m+1}$ be 1-1. There are two possibilities for $f(n)$.

Case 1: Let $f(n) = m + 1$. Then consider the map $g : I_{n-1} \to I_m$ given by $g(j) = f(j)$. Then $g$ is 1-1 and hence $I_m$ is not true.

Case 2: Let $f(n) = r < m+1$. Then there is at most one $1 \leq k < n$ such that $f(k) = m+1$. We define $g : I_{n-1} \to I_m$ by setting $g(j) = f(j)$ for $j \neq k$ and $g(k) = r = f(n)$. Then $g$ is 1-1 and hence $P(m)$ is not true.

Thus we conclude that such an $f$ cannot exist. In other words, $P(m+1)$ is also true. By the principle of induction, we conclude that $P(m)$ is true for all $m$ and hence the lemma is proved. □

**Lemma 3.** *If $m < n$, then there is no onto map $f : I_m \to I_n$.*

*Proof.* Let $f : I_m \to I_n$ be onto where $m < n$. We define $g : I_n \to I_m$ as follows: Let $r \in I_n$. Let $i := \min f^{-1}(r)$. We set $g(r) = i$. Then $g$ is 1-1: If $g(r) = g(s)$, then there exists $k \in I_m$ such that $f(k) = r, s$, i.e., $f$ is not a function! □

**Corollary 4.** *If $f : I_m \to I_n$ is a bijection, then $m = n$.*

**Definition 5.** A finite set $A$ is said to have $n$ elements, if there is a bijection $f : A \to I_n$. Note that in view of the corollary this is well-defined. For any finite set $A$, we let $|A|$ denote the number of elements in $A$.

**Lemma 6.** *Let $f\colon A \to I_n$ be 1-1. Then $A$ is finite, and we have $|A| \le n$.*

*Proof.* Let $r_1 = \min\{f(a) : a \in A\} \subset \mathbb{N}$, $r_2 = \min\{f(A) \setminus \{r_1\}\}$. Note that $r_1 \ge 1$ and $r_2 > r_1$ so that $r_2 \ge 2$. We proceed by induction to construct $r_1 < r_2 < \cdots < r_k$ where $r_k \ge k$. This process will stop at some stage in the sense that $f(A) \setminus \{r_j : 1 \le j \le k\} = \emptyset$ for some $k \le n$. For, otherwise, if $k > n$, then $r_k \ge k > n$. This contradicts the fact that $r_k \in I_n$. We now construct a bijection $g\colon I_k \to A$ as follows: $g(i) = a$ where $f(a) = r_i$. One easily checks that $g$ is a bijection. □

**Corollary 7.** *If $A$ is finite and $B \subset A$, then $B$ is finite and $|B| \le |A|$.*

*Proof.* Let $f\colon A \to I_n$ be a bijection. Then the composition of the inclusion $B \hookrightarrow A$ followed by $f$ is a 1-1 map of $B$ into $I_n$. By Lemma 6, the result follows. □

**Lemma 8.** *Let $f\colon I_n \to A$ be onto. Then $A$ is finite and $|A| \le n$.*

*Proof.* Define $g\colon A \to I_n$ by setting $g(a) = \min f^{-1}(a)$. Then $g$ is 1-1 and the result follows from the last corollary. □

**Proposition 9** (Pigeonhole Principle)**.** *Let $m, n \in \mathbb{N}$ be such that $m < n$. If $f\colon I_n \to I_m$ is a map, then there exists $i, j \in I_n$ such that $i \ne j$ and $f(i) = f(j)$.* □