Plancherel Theorem and Fourier Inversion Theorem

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1 Plancherel Theorem

Let $f \in L^1(\mathbb{R})$. We define the Fourier transform $\mathcal{F}f(x) := \hat{f}(x) := \int_{\mathbb{R}} f(t)e^{-ixt}\,dt$ for $x \in \mathbb{R}$. The main results of these lectures are the **Plancherel theorem** which states that the linear map $\mathcal{F} : L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \to L^2(\mathbb{R})$ extends to an "isometry" of $L^2(\mathbb{R})$ onto itself and that for a continuous $f \in L^1(\mathbb{R})$ with $\hat{f} \in L^1(\mathbb{R})$ we have the **Fourier inversion formula:**

$$f(x) = \int_{\mathbb{R}} \hat{f}(y)e^{iyx}\,dy \text{ for all } x \in \mathbb{R}.$$

We recall first the definition of a step function. A function $f : \mathbb{R} \to \mathbb{C}$ is called a step function if it is a finite linear combination of characteristic (or indicator) functions of finite intervals. Recall also that $\mathcal{S}$, the space of step functions is dense in $L^p(\mathbb{R})$ for all $1 \leq p < \infty$. Thus it is natural to verify the assertion of the Plancherel theorem in the case of $f := \mathbf{1}_J$, the indicator function of a finite interval $J = (a,b), [a,b), [a,b], (a,b]$. Notice that whatever be the form of J, the indicator functions $\mathbf{1}_J$ are all the same as elements of L^p. We now compute the Fourier transform of f:

$$\hat{f}(x) = \int_{\mathbb{R}} f(t)e^{-ixt}dt = \int_a^b e^{-ixt}dt = \frac{e^{-ibx} - e^{-iax}}{-ix}.$$

Also, we have

$$\| f \|_2^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_a^b |1|^2 dt = \int_a^b 1 dt = b - a.$$

We now check whether $\hat{f}$ lies in $L^2(\mathbb{R})$ and if so, compute its norm. Here we go:

$$\begin{aligned} |\hat{f}(x)|^2 &= \left|\frac{e^{-ibx} - e^{-iax}}{-ix}\right|^2 = \frac{e^{-ibx} - e^{-iax}}{-ix}\overline{\left(\frac{e^{-ibx} - e^{-iax}}{-ix}\right)} \\ &= \frac{e^{-ibx} - e^{-iax}}{-ix}\frac{e^{ibx} - e^{iax}}{ix} = \frac{2 - (e^{-i(b-a)x} + e^{i(b-a)x})}{x^2} \\ &= \frac{2 - 2\cos(b-a)x}{x^2} = 2\frac{1 - \cos(b-a)x}{x^2} \\ &= 2 \cdot 2\frac{\sin^2((b-a)x/2)}{x^2}, \end{aligned}$$

where in the last we have used a well-known trigonometric identity. We thus find:

$$\left\| \hat{f}(x) \right\|_2^2 = \int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx = 4 \int_{-\infty}^{\infty} \frac{\sin^2((b-a)x/2)}{x^2} dx.$$

We put $u := (b-a)x/2$ so that the above becomes

$$\begin{aligned} \left\| \hat{f}(x) \right\|_2^2 &= 4 \int_{\mathbb{R}} \frac{\sin^2 u}{4u^2} (b-a)^2 \frac{2\,du}{b-a} \\ &= 2 \int_{\mathbb{R}} \frac{\sin^2 u}{u^2} (b-a)\, du \\ &= C(b-a). \end{aligned}$$

Here we have let C stand for the *real number* $2 \int_{\mathbb{R}} \frac{\sin^2 u}{u^2} du$.

Remark 1. Let us observe that C is a (finite) real number, i.e., $\sin^2 u/u^2$ is integrable on $\mathbb{R}$. For, the function $g(x) := \begin{cases} \sin x/x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ is continuous on $\mathbb{R}$. Hence the continuous function $\sin^2 x/x^2$ is integrable over the finite interval $[-1,1]$ and it is dominated by the continuous function $1/x^2$ on $\mathbb{R} \setminus [-1,1]$ on which $1/x^2$ is integrable. Hence C is finite.

We now define $\mathcal{F}^*$ on $\mathcal{S}$ as follows:

$$\mathcal{F}^* f(x) := \int_{\mathbb{R}} f(t) e^{ixt}\, dt, \qquad \text{for } f \in \mathcal{S}.$$

Proceeding as above, we find that $\| \mathcal{F}^* f \|_2^2 = C \| f \|_2^2$ for $f := \mathbf{1}_{(a,b)}$, for the same C.

If $f := \mathbf{1}_{(a,b)}$ and $g := \mathbf{1}_{(c,d)}$, then we have:

$$\langle \mathcal{F} f, g \rangle = \int_c^d [\int_a^b e^{-ixt} dt] 1 dx = \int_a^b 1 \overline{[\int_c^d e^{ixt} dx]}\, dt = \langle f, \mathcal{F}^* g \rangle .$$

Thus on the indicator functions, $\mathcal{F}^*$ behaves like the adjoint of $\mathcal{F}$. We now wish to extend these results to $f, g \in \mathcal{S}$. We observe that if $f := \mathbf{1}_{(a,b)}$, and $g := \mathbf{1}_{(b,c)}$, then

$$\begin{aligned} (\mathcal{F} f + \mathcal{F} g)(x) &= \hat{f}(x) + \hat{g}(x) = \int_a^b e^{-ixt} dt + \int_b^c e^{-ixt} dt \\ &= \int_a^c e^{-ixt} dt = \mathcal{F}(f+g)(x). \end{aligned}$$

Hence it follows that

$$\| \mathcal{F} f + \mathcal{F} g \|_2^2 = \| \mathcal{F}(f+g) \|_2^2 = \int_{\mathbb{R}} | \int_a^c e^{-ixt}\, dt |^2 dx = C(c-a).$$

But, since,

$$\| \mathcal{F} f + \mathcal{F} g \|_2^2 = \langle \mathcal{F} f + \mathcal{F} g, \mathcal{F} f + \mathcal{F} g \rangle = \| \mathcal{F} f \|_2^2 + \| \mathcal{F} g \|_2^2 + 2 \cdot \mathrm{Re} \left\langle \hat{f}, \hat{g} \right\rangle ,$$

we find that Re $\left\langle \hat{f}, \hat{g} \right\rangle = 0$, for f and g as above. Similar result holds true also for $\mathcal{F}^*$.

Even if $f := \mathbf{1}_{(a,b)}$ and $g := \mathbf{1}_{(c,d)}$ with $a \leq b < c \leq d$ we have Re $\left\langle \hat{f}, \hat{g} \right\rangle = 0$. To see this, let $h := \mathbf{1}_{(b,c)}$. Then using the earlier result, we have Re $\left\langle \hat{f} + \hat{h}, \hat{g} \right\rangle = 0$ and Re $\left\langle \hat{h}, \hat{g} \right\rangle = 0$. Subtracting the latter from the first, we get Re $\left\langle \hat{f}, \hat{g} \right\rangle = 0$. Similarly for $\mathcal{F}^*$.

We note that $\mathcal{F}\mathbf{1}_{(a,b)}$ satisfies: $\mathcal{F}\mathbf{1}_{(a,b)}(-x) = \overline{\mathcal{F}\mathbf{1}_{(a,b)}(x)}$, i.e., $\hat{f}(-x) = \overline{\hat{f}(x)}$:

$$\hat{f}(-x) = \int_a^b e^{ixt} dt = \overline{\int_a^b e^{-ixt} dt} = \overline{\hat{f}(x)}.$$

If $f, g \in L^2(\mathbb{R})$ satisfy $(*)$, i.e., $f(-x) = \overline{f(x)}$ etc., then we have

$$\begin{aligned} \overline{\langle f, g \rangle} &= \overline{\int f(x)\overline{g(x)} dx} = \int \overline{f(x)} g(x) dx = \int f(-x) \overline{g(-x)} dx \\ &= \int f(x) \overline{g(x)} dx = \langle f, g \rangle . \end{aligned}$$

where we have used the fact that the Lebesgue measure is invariant under $x \mapsto -x$. This observation, when applied to $\hat{f}$ and $\hat{g}$ for $f := \mathbf{1}_{(a,b)}$ and $g := \mathbf{1}_{(c,d)}$, allows us to conclude $\left\langle \hat{f}, \hat{g} \right\rangle = 0$ and $\langle \mathcal{F}f, \mathcal{F}g \rangle = 0$. That is, we can drop the prefix "Re" in Re $\left\langle \hat{f}, \hat{g} \right\rangle$.

Now if f is any step function, say, of the form $f = \sum_{i=1}^n a_i \mathbf{1}_{J_i}$ where J_i are finite intervals and $a_i \in \mathbb{C}$, we can write $f = \sum_{j-1}^N b_j \mathbf{1}_{I_j}$, where I_j are pair-wise disjoint finite intervals. (It is easier to convince yourself of this than writing down a formal verbose proof!) We then have

$$\begin{aligned} \| \mathcal{F}f \|_2^2 &= \left\langle \sum_j b_j \mathcal{F}\mathbf{1}_{I_j}, \sum_k b_k \mathcal{F}\mathbf{1}_{I_k} \right\rangle \\ &= \sum_{j,k} b_j \overline{b_k} C \left\langle \mathbf{1}_{I_j}, \mathbf{1}_{I_k} \right\rangle \\ &= C \sum |b_j|^2 \left\| \mathbf{1}_{I_j} \right\|^2 = C \int |f(x)|^2 \, dx = C \| f \|_2^2 . \end{aligned}$$

Similarly, we have $\| \mathcal{F}^* f \|_2^2 = C \| f \|_2^2$, for $f \in \mathcal{S}$. Also, by the bilinearity of the inner product we have

$$\langle \mathcal{F}f, g \rangle = \langle f, \mathcal{F}^* g \rangle , \qquad \text{for } f, g \in \mathcal{S}.$$

Thus we have linear maps $\mathcal{F}, \mathcal{F}^* : \mathcal{S} \to L^2(\mathbb{R})$ such that i) $\| \mathcal{F}f \|_2^2 = C \| f \|_2^2 = \| \mathcal{F}^* f \|_2^2$, and ii) $\langle \mathcal{F}f, g \rangle = \langle f, \mathcal{F}^* g \rangle$ for all $f, g \in \mathcal{S}$. Since $\mathcal{S}$ is dense in $L^2(\mathbb{R})$ and $\mathcal{F}, \mathcal{F}^*$ are continuous linear, we have unique extensions, denoted again by $\mathcal{F}$ and $\mathcal{F}^*$, from $L^2(\mathbb{R})$ to itself. This follows from the following elementary result:

Lemma 2. *Let $T : D \subset E \to F$ be a continuous linear map defined on a dense subspace D of E to a Banach space F. Then T has a unique continuous linear extension $\overline{T} : E \to F$ such that $\left\| \overline{T} \right\|_{(E,F)} = \| T \|_{(D,F)}$ (operator norms).*

Proof. We shall only sketch the proof.

For $x \in E$, take any $x_n \in D$ such that $\| x - x_n \| \to 0$. Then define $\overline{T}(x) := \lim T x_n$ which exists since $T x_n$ is Cauchy in F (due to the uniform continuity of a continuous linear map!). If $y_n \in D$ is such that $\| y_n - x \| \to 0$ then it can be easily seen that $\lim T y_n = \lim T x_n$ so that $\overline{T}x$ is well defined. $\square$

Hence we have $\| \mathcal{F} f \|_2^2 = C \| f \|_2^2 = \| \mathcal{F}^* f \|_2^2$ and $\langle \mathcal{F} f, g \rangle = \langle f, \mathcal{F}^* g \rangle$ for all $f, g \in L^2(\mathbb{R})$, by continuity of the inner product.

We also have

$$\begin{aligned}
\langle f, g \rangle &= \frac{1}{4}[\| f + g \|^2 + i \| f + ig \|^2 - \| f - g \|^2 - i \| f - ig \|^2] \\
&= \frac{1}{4C}[\| \mathcal{F} f + \mathcal{F} g \|^2 + i \| \mathcal{F} f + i \mathcal{F} g \|^2 - \| \mathcal{F} f - \mathcal{F} g \|^2 - i \| \mathcal{F} f - i \mathcal{F} g \|^2] \\
&= \frac{1}{C} \langle \mathcal{F} f, \mathcal{F} g \rangle = \frac{1}{C} \langle \mathcal{F}^* \mathcal{F} f, g \rangle .
\end{aligned}$$

The last equality is valid, as $\langle h, \mathcal{F} g \rangle = \langle \mathcal{F}^* h, g \rangle$ where $h = \mathcal{F} f \in L^2(\mathbb{R})$.

We put $g := \mathcal{F}^* \mathcal{F} f - f \in L^2(\mathbb{R})$ in $\langle f, g \rangle = (1/C) \langle \mathcal{F}^* \mathcal{F} f, g \rangle$ to get

$$0 = \left\langle f - \frac{1}{C} \mathcal{F}^* \mathcal{F} f, g \right\rangle = \left\| f - \frac{1}{C} \mathcal{F}^* \mathcal{F} f \right\|^2 = 0.$$

That is, $\mathcal{F}^* \mathcal{F} f = C f$ a.e. Similarly, $\mathcal{F} \mathcal{F}^* f = C f$ a.e. Thus we have proved the following theorem:

Theorem 3 (Plancherel)**.** *Let $\mathcal{S}$ denote the dense subspace of the step functions in $L^2(\mathbb{R})$. Let $\mathcal{F}, \mathcal{F}^*$ denote the Fourier and conjugate Fourier transforms defined as above. Then, for C as above,*

$\mathcal{F}$ and $\mathcal{F}^*$ map $\mathcal{S}$ into $L^2(\mathbb{R})$; in fact, we have:

$$\| \mathcal{F} f \|^2 = C \| f \|^2 = \| \mathcal{F}^* f \|^2 \qquad \text{for } f \in \mathcal{S}.$$

$\mathcal{F}, \mathcal{F}^*$ extend to an "isometry" of $L^2(\mathbb{R})$ onto itself; that is, $\mathcal{F} \mathcal{F}^* = C = \mathcal{F}^* \mathcal{F}$ on $L^2(\mathbb{R})$. $\square$

2 Fourier Inversion Theorem

We may ask whether for $f \in L^1(\mathbb{R})$ we have the formula $\hat{f}(x) = \int_{\mathbb{R}} f(t) e^{-ixt}\, dt$ and motivated by the Plancherel theorem whether for nice enough functions we can invert the Fourier transform, i.e., $f(t) = \int \hat{f}(x) e^{ixt}\, dx$.

The first formula is not all obvious even if we assume that $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, as we have extended $\mathcal{F}$ to $L^2(\mathbb{R})$ by an abstract procedure. However, this is easy to justify: We start with a non-negative $f \in L^1(\mathbb{R})$ and take any sequence f_n of step functions increasing to f. We can now apply the monotone convergence theorem to conclude that $\mathcal{F} f$ is given by the above formula.

The proof of the second is given as the conclusion of the following theorem:

Theorem 4 (Fourier Inversion Theorem)**.** *Let f be a continuous function in $L^1(\mathbb{R})$. Assume that $\hat{f} \in L^1(\mathbb{R})$. Then we have*

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(y)e^{iyt}\,dy, \qquad \text{for all } x \in \mathbb{R}.$$

Proof. The double integral $\int \hat{f}(y)e^{ixy}dx = \int(\int_{\mathbb{R}} f(t)e^{-iyt}\,dt)e^{ixy}\,dx$ may not be absolutely convergent (the trouble lies in the x-variable) and hence we cannot use Fubini to evaluate it as an iterated integral. So, what we do, is to adopt a classical trick of introducing a convergence factor in the x-variable. We take a "nice" function ψ such as a continuous function with compact support with $\hat{\psi} \in L^1(\mathbb{R})$, or $\psi(y) := e^{-y^2}$ or any function that "decays rapidly at ∞") with $\psi(0) = 1$. If you wish you may take $\psi(y) = e^{-y^2}$ in the following.

We have by dominated convergence theorem

$$\lim_{\varepsilon\to 0}\int \psi(\varepsilon y)\hat{f}(y)e^{ixy}\,dy = \int \hat{f}(y)e^{ixy}\,dy. \tag{1}$$

We unwind Eq. 1 and use Fubini on the LHS (of Eq. 1):

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int \psi(\varepsilon y)\hat{f}(y)e^{ixy}\,dy &= \lim_{\varepsilon\to 0}\int \psi(\varepsilon y)(\int_{\mathbb{R}} f(t)e^{-iyt}\,dt)e^{ixy}\,dy \\
&= \lim_{\varepsilon\to 0}\int f(t)(\int \psi(\varepsilon y)e^{-iy(t-x)}\,dy)\,dt \\
&= \lim_{\varepsilon\to 0}\int f(t)(\int \psi(u)e^{-\frac{iu}{\varepsilon}(t-x)}\,du)\,dt \qquad \text{where } u = \varepsilon y \\
&= \lim_{\varepsilon\to 0}\int f(t)\hat{\psi}(\frac{t-x}{\varepsilon})\,\frac{dt}{\varepsilon} \\
&= \lim_{\varepsilon\to 0}\int f(x+\varepsilon v)\hat{\psi}(v)\,dv \qquad \text{where } \varepsilon v = t - x \\
&= f(x)\int \hat{\psi}(v)dv,
\end{aligned}$$

the last equality being in view of the continuity of f and dominated convergence theorem. This completes the proof of the theorem, except for an irritating but minor detail to be attended to. For some ψ we need to compute $\int \hat{\psi}(v)\,dv$, which in view of the conclusion of the theorem should be nothing other than a constant times $\psi(0)$. By computing the Fourier transform of e^{-x^2}, we can have satisfaction. □

Remark 5. It is traditional to derive the Plancherel theorem from the Fourier inversion theorem as follows:

Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Take $g(t) := \overline{f(-t)}$. Then, $f * g$ is continuous and lies in $L^1(\mathbb{R})$. We have by the definition of convolution

$$f * g(0) = \int f(-t)g(t)\,dt = \int f(-t)\overline{f(-t)}\,dt = \| f \|_2^2 .$$

On the other hand, by the inversion formula, we have

$$f * g(0) = C\int \widehat{f * g}(x)\,dx = C\int \hat{f}(x)\hat{g}(x)\,dx = C\int \hat{f}(x)\overline{\hat{f}(x)}\,dx = C\left\|\hat{f}\right\|_2^2 .$$

The Plancherel theorem follows from Eq. 5 and Eq. 5.

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