1 Plancherel Theorem

Let \( f \in L^1(\mathbb{R}) \). We define the Fourier transform \( \mathcal{F}f(x) := \hat{f}(x) := \int_{\mathbb{R}} f(t)e^{-ixt} \, dt \) for \( x \in \mathbb{R} \).

The main results of these lectures are the \textit{Plancherel theorem} which states that the linear map \( \mathcal{F} : L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) extends to an "isometry" of \( L^2(\mathbb{R}) \) onto itself and that for a continuous \( f \in L^1(\mathbb{R}) \) with \( \hat{f} \in L^1(\mathbb{R}) \) we have the \textit{Fourier inversion formula}:

\[
\hat{f}(x) = \int_{\mathbb{R}} \hat{f}(y)e^{iyx} \, dy \text{ for all } x \in \mathbb{R}.
\]

We now check whether \( \hat{f} \) lies in \( L^2(\mathbb{R}) \) and if so, compute its norm. Here we go:

\[
|\hat{f}(x)|^2 = \left| \frac{e^{-ibx} - e^{-iax}}{-ix} \right|^2 = \frac{e^{-ibx} - e^{-iax} (e^{-ibx} - e^{-iax})}{-ix} = \frac{e^{-ibx} - e^{-iax} e^{ibx} - e^{iax}}{-ix} = 2 - (e^{-i(b-a)x} + e^{i(b-a)x}) \]
\[
= \frac{2 - 2 \cos(b-a)x}{x^2} = \frac{2(1 - \cos(b-a)x)}{x^2} = 2 \cdot \frac{2 \sin^2((b-a)x/2)}{x^2},
\]
where in the last we have used a well-known trigonometric identity. We thus find:

\[ \left\| \hat{f}(x) \right\|_2^2 = \int_{-\infty}^{\infty} |\hat{f}(x)|^2 \, dx = 4 \int_{-\infty}^{\infty} \frac{\sin^2((b-a)x/2)}{x^2} \, dx. \]

We put \( u := (b-a)x/2 \) so that the above becomes

\[
\left\| \hat{f}(x) \right\|_2^2 = 4 \int_{R} \frac{\sin^2 u (b-a)^2}{4u^2} \frac{2 \, du}{b-a} = 2 \int_{R} \frac{\sin^2 u (b-a)}{u^2} \, du = C(b-a).
\]

Here we have let \( C \) stand for the real number \( 2 \int_{R} \frac{\sin^2 u}{u^2} \, du \).

**Remark 1.** Let us observe that \( C \) is a (finite) real number, i.e., \( \sin^2 u/u^2 \) is integrable on \( R \). For, the function \( g(x) := \begin{cases} \sin x/x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases} \) is continuous on \( R \). Hence the continuous function \( \sin^2 x/x^2 \) is integrable over the finite interval \([-1, 1]\) and it is dominated by the continuous function \( 1/x^2 \) on \( R \setminus [-1, 1] \) on which \( 1/x^2 \) is integrable. Hence \( C \) is finite.

We now define \( F^* \) on \( S \) as follows:

\[ F^* f(x) := \int_{R} f(t)e^{ixt} \, dt, \quad \text{for } f \in S. \]

Proceeding as above, we find that \( \| F^* f \|_2^2 = C \| f \|_2^2 \) for \( f := 1_{(a,b)} \), for the same \( C \).

If \( f := 1_{(a,b)} \) and \( g := 1_{(c,d)} \), then we have:

\[ \langle F f, g \rangle = \int_{c}^{d} \left[ \int_{a}^{b} e^{-ixt} \, dt \right] \, dx = \int_{c}^{d} \left[ 1_{\left( a, \frac{b}{2} \right]} e^{ixt} \, dx \right] \, dt = \langle f, F^* g \rangle. \]

Thus on the indicator functions, \( F^* \) behaves like the adjoint of \( F \). We now wish to extend these results to \( f, g \in S \). We observe that if \( f := 1_{(a,b)} \), and \( g := 1_{(b,c)} \), then

\[
(F f + F g)(x) = \hat{f}(x) + \hat{g}(x) = \int_{a}^{b} e^{-ixt} \, dt + \int_{b}^{c} e^{-ixt} \, dt = \int_{a}^{c} e^{-ixt} \, dt = F(f + g)(x).
\]

Hence it follows that

\[ \| F f + F g \|_2^2 = \| F(f + g) \|_2^2 = \int_{R} \left| \int_{a}^{c} e^{-ixt} \, dt \right|^2 \, dx = C(c-a). \]

But, since,

\[ \| F f + F g \|_2^2 = \langle F f + F g, F f + F g \rangle = \| F f \|_2^2 + \| F g \|_2^2 + 2 \cdot \text{Re} \left( \hat{f}, \hat{g} \right), \]
we find that $\text{Re} \left( \hat{f}, \hat{g} \right) = 0$, for $f$ and $g$ as above. Similar result holds true also for $\mathcal{F}^*$.

Even if $f := 1_{(a,b)}$ and $g := 1_{(c,d)}$ with $a \leq b < c \leq d$ we have $\text{Re} \left( \hat{f}, \hat{g} \right) = 0$. To see this, let $h := 1_{(b,c)}$. Then using the earlier result, we have $\text{Re} \left( \hat{f} + \hat{h}, \hat{g} \right) = 0$ and $\text{Re} \left( \hat{h}, \hat{g} \right) = 0$. Subtracting the latter from the first, we get $\text{Re} \left( \hat{f}, \hat{g} \right) = 0$. Similarly for $\mathcal{F}^*$.

We note that $\mathcal{F} 1_{(a,b)}$ satisfies: $\mathcal{F} 1_{(a,b)}(-x) = \overline{\mathcal{F} 1_{(a,b)}(x)}$, i.e., $\hat{f}(-x) = \overline{\hat{f}(x)}$:

$$\hat{f}(-x) = \int_a^b e^{ixt} dt = \overline{\int_a^b e^{-ixt} dt} = \overline{\hat{f}(x)}.$$  

If $f, g \in L^2(\mathbb{R})$ satisfy (*), i.e., $f(-x) = \overline{f(x)}$ etc., then we have

$$\langle f, g \rangle = \int f(x) \overline{g(x)} dx = \int f(x) g(x) dx = \int f(-x) g(-x) dx = \int f(x) \overline{g(x)} dx = \langle f, g \rangle.$$  

where we have used the fact that the Lebesgue measure is invariant under $x \mapsto -x$. This observation, when applied to $\hat{f}$ and $\hat{g}$ for $f := 1_{(a,b)}$ and $g := 1_{(c,d)}$, allows us to conclude $\langle \hat{f}, \hat{g} \rangle = 0$ and $\langle \mathcal{F} f, \mathcal{F} g \rangle = 0$. That is, we can drop the prefix “Re” in $\text{Re} \left( \hat{f}, \hat{g} \right)$.

Now if $f$ is any step function, say, of the form $f = \sum_{i=1}^n a_i 1_{J_i}$ where $J_i$ are finite intervals and $a_i \in \mathbb{C}$, we can write $f = \sum_{j=1}^N b_j 1_{I_j}$, where $I_j$ are pair-wise disjoint finite intervals. (It is easier to convince yourself of this than writing down a formal verbose proof!) We then have

$$\| \mathcal{F} f \|_2^2 = \left\langle \sum_j b_j \mathcal{F} 1_{I_j}, \sum_k b_k \mathcal{F} 1_{I_k} \right\rangle = \sum_{j,k} b_j \overline{b_k} C \langle 1_{I_j}, 1_{I_k} \rangle = C \sum |b_j|^2 \| 1_{I_j} \|^2 = C \int |f(x)|^2 dx = C \| f \|_2^2.$$  

Similarly, we have $\| \mathcal{F}^* f \|_2^2 = C \| f \|_2^2$, for $f \in \mathcal{S}$. Also, by the bilinearity of the inner product we have

$$\langle \mathcal{F} f, g \rangle = \langle f, \mathcal{F}^* g \rangle,$$  

for $f, g \in \mathcal{S}$.

Thus we have linear maps $\mathcal{F}, \mathcal{F}^* : \mathcal{S} \to L^2(\mathbb{R})$ such that i) $\| \mathcal{F} f \|_2^2 = C \| f \|_2^2 = \| \mathcal{F}^* f \|_2^2$, and ii) $\langle \mathcal{F} f, g \rangle = \langle f, \mathcal{F}^* g \rangle$ for all $f, g \in \mathcal{S}$. Since $\mathcal{S}$ is dense in $L^2(\mathbb{R})$ and $\mathcal{F}, \mathcal{F}^*$ are continuous linear, we have unique extensions, denoted again by $\mathcal{F}$ and $\mathcal{F}^*$, from $L^2(\mathbb{R})$ to itself. This follows from the following elementary result:

**Lemma 2.** Let $T : D \subset E \to F$ be a continuous linear map defined on a dense subspace $D$ of $E$ to a Banach space $F$. Then $T$ has a unique continuous linear extension $\overline{T} : E \to F$ such that $\| \overline{T} \|_{(E,F)} = \| T \|_{(D,F)}$ (operator norms).
Proof. We shall only sketch the proof.

For $x \in E$, take any $x_n \in D$ such that $\|x - x_n\| \to 0$. Then define $T(x) := \lim Tx_n$ which exists since $Tx_n$ is Cauchy in $F$ (due to the uniform continuity of a continuous linear map!). If $y_n \in D$ is such that $\|y_n - x\| \to 0$ then it can be easily seen that $\lim Ty_n = \lim Tx_n$ so that $Tx$ is well defined.

Hence we have $\|Ff\|_2^2 = C \|f\|_2^2 = \|F^*f\|_2^2$ and $\langle Ff, g \rangle = \langle f, F^*g \rangle$ for all $f, g \in L^2(\mathbb{R})$, by continuity of the inner product.

We also have

\[ \langle f, g \rangle = \frac{1}{4} \left[ \|f + g\|^2 + i \|f + ig\|^2 - \|f - g\|^2 - i \|f - ig\|^2 \right] \]

\[ = \frac{1}{4C} \left[ \|Ff + Fg\|^2 + i \|Ff + iFg\|^2 - \|Ff - Fg\|^2 - i \|Ff - iFg\|^2 \right] \]

\[ = \frac{1}{C} \langle Ff, Fg \rangle = \frac{1}{C} \langle F^*Ff, g \rangle. \]

The last equality is valid, as $\langle h, Fg \rangle = \langle F^*h, g \rangle$ where $h = Ff \in L^2(\mathbb{R})$.

We put $g := F^*Ff - f \in L^2(\mathbb{R})$ in $\langle f, g \rangle = (1/C) \langle F^*Ff, g \rangle$ to get

\[ 0 = \left\langle f - \frac{1}{C} F^*Ff, g \right\rangle = \left\| f - \frac{1}{C} F^*Ff \right\|^2 = 0. \]

That is, $F^*Ff = Cf$ a.e. Similarly, $F^*Ff = Cf$ a.e. Thus we have proved the following theorem:

**Theorem 3** (Plancherel). Let $S$ denote the dense subspace of the step functions in $L^2(\mathbb{R})$. Let $F, F^*$ denote the Fourier and conjugate Fourier transforms defined as above. Then, for $C$ as above,

$F$ and $F^*$ map $S$ into $L^2(\mathbb{R})$; in fact, we have:

$\|Ff\|^2 = C \|f\|^2 = \|F^*f\|^2$ for $f \in S$.

$F, F^*$ extend to an “isometry” of $L^2(\mathbb{R})$ onto itself; that is, $F^*F = C = F^*F$ on $L^2(\mathbb{R})$. □

2 Fourier Inversion Theorem

We may ask whether for $f \in L^1(\mathbb{R})$ we have the formula $\hat{f}(x) = \int_{\mathbb{R}} f(t)e^{-ixt} dt$ and motivated by the Plancherel theorem whether for nice enough functions we can invert the Fourier transform, i.e., $f(t) = \int \hat{f}(x)e^{ixt} dx$.

The first formula is not all obvious even if we assume that $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, as we have extended $F$ to $L^2(\mathbb{R})$ by an abstract procedure. However, this is easy to justify: We start with a non-negative $f \in L^1(\mathbb{R})$ and take any sequence $f_n$ of step functions increasing to $f$.

We can now apply the monotone convergence theorem to conclude that $Ff$ is given by the above formula.

The proof of the second is given as the conclusion of the following theorem:
Theorem 4 (Fourier Inversion Theorem). Let $f$ be a continuous function in $L^1(\mathbb{R})$. Assume that $\hat{f} \in L^1(\mathbb{R})$. Then we have

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(y)e^{ixy} dy, \quad \text{for all } x \in \mathbb{R}. $$

Proof. The double integral $\int \hat{f}(y)e^{ixy} dy = \int (\int_{\mathbb{R}} f(t)e^{-ity} dt)e^{ixy} dx$ may not be absolutely convergent (the trouble lies in the $x$-variable) and hence we cannot use Fubini to evaluate it as an iterated integral. So, what we do, is to adopt a classical trick of introducing a convergence factor in the $x$-variable. We take a “nice” function $\psi$ such as a continuous function with compact support with $\hat{\psi} \in L^1(\mathbb{R})$, or $\psi(y) := e^{-y^2}$ or any function that “decays rapidly at $\infty$”) with $\psi(0) = 1$. If you wish you may take $\psi(y) = e^{-y^2}$ in the following.

We have by dominated convergence theorem

$$\lim_{\varepsilon \to 0} \int \psi(\varepsilon y)\hat{f}(y)e^{ixy} dy = \int \hat{f}(y)e^{ixy} dy. \quad (1)$$

We unwind Eq. 1 and use Fubini on the LHS (of Eq. 1):

$$\lim_{\varepsilon \to 0} \int \psi(\varepsilon y)\hat{f}(y)e^{ixy} dy = \lim_{\varepsilon \to 0} \int \psi(\varepsilon y)(\int_{\mathbb{R}} f(t)e^{-ity} dt)e^{ixy} dy$$

$$= \lim_{\varepsilon \to 0} \int f(t)(\int \psi(\varepsilon y)e^{-ity} dy) dt$$

$$= \lim_{\varepsilon \to 0} \int f(t)(\int \psi(u)e^{-iu(t-x)} du) dt \quad \text{where } u = \varepsilon y$$

$$= \lim_{\varepsilon \to 0} \int f(t)\hat{\psi}(\frac{t-x}{\varepsilon}) \frac{dt}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \int f(x+\varepsilon v)\hat{\psi}(v) dv \quad \text{where } \varepsilon v = t - x$$

$$= f(x) \int \hat{\psi}(v) dv,$$

the last equality being in view of the continuity of $f$ and dominated convergence theorem. This completes the proof of the theorem, except for an irritating but minor detail to be attended to. For some $\psi$ we need to compute $\int \hat{\psi}(v) dv$, which in view of the conclusion of the theorem should be nothing other than a constant times $\hat{\psi}(0)$. By computing the Fourier transform of $e^{-x^2}$, we can have satisfaction. \hfill \Box

Remark 5. It is traditional to derive the Plancherel theorem from the Fourier inversion theorem as follows:

Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Take $g(t) := \overline{f(-t)}$. Then, $f \ast g$ is continuous and lies in $L^1(\mathbb{R})$. We have by the definition of convolution

$$f \ast g(0) = \int f(-t)g(t) dt = \int f(-t)\overline{f(-t)} dt = \|f\|_2^2.$$ 

On the other hand, by the inversion formula, we have

$$f \ast g(0) = C \int f \ast g(x) dx = C \int \hat{f}(x)\hat{g}(x) dx = C \int \hat{f}(x)\overline{\hat{f}(x)} dx = C \|\hat{f}\|_2^2.$$ 

The Plancherel theorem follows from Eq. 5 and Eq. 5.
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