Frenet-Serre Formulas for Space Curves

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1 Introduction

I wrote this article to introduce the readers to a modern outlook of Serre-Frenet Formulas via the Maurer-Cartan forms. While the treatment is hetrogeneous, a diligent reader can cull an elementary introduction as well as a more advanced perspective of these formulas.

One of the basic ideas in differential geometry is to introduce at each point of the geometric quantity (in the case on hand, a curve) a suitable orthonormal basis of the underlying Euclidean space. This procedure should be compared with the change of coordinates with which the reader may be acquainted in the study of conics. There the idea was to introduce a suitable basis which once and for all transforms the equation of the given conic to a simpler form from which geometric results are easy to obtain.

If a curve c is given, we are looking for an orthonormal basis $\{e_1(s), e_2(s), e_3(s)\}$ for each $s \in (a, b)$ having the same orientation as the standard basis. The last condition means that

$$\det(e_1(s), e_2(s), e_3(s)) = 1$$

if we think of $e_i(s)$ as a column vector with respect to the standard basis. If we set $\mathbf{e}_i'(s) = \sum_j \omega_{ji} e_j(s)$, then by Ex. 1, we have $\omega_{ji}(s) + \omega_{ij}(s) = 0$ for all $1 \le i, j \le 3$. Thus we get a skew-symmetric matrix

$$\Omega := \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix}.$$

The idea of the Frenet type frames is to choose $e_i(s)$ in such a way that Ω becomes simpler. We may take $e_1(s) := \mathbf{t}(s)$, the unit tangent vector. Thus we need to choose an orthonormal basis $\{e_2^*(s), e_3^*(s)\}$ of $(\mathbb{R}c'(s))^\perp$, the two dimensional orthogonal complement of $\mathbf{t}$.

We derive the Serret-Frenet formulas for the space curves analogous to the plane curves. Here also our basic idea is to find an orthonormal frame $\{e_1(s), e_2(s), e_3(s)\}$ at each point $c(s)$ of c having the same orientation as the standard basis $\{e_1, e_2, e_3\}$.

Ex. 1. This exercise is at the heart of most of the computations below.

i) A *vector field* $\mathbf{v}$ along a curve $c\colon (a, b) \to \mathbb{R}^n$ is a map $\mathbf{v}\colon (a, b) \to \mathbb{R}^n$. Then $\mathbf{v}(s)$ is thought of as a vector at $c(s)$. We say that $s \mapsto \mathbf{v}(s)$ is differentiable if the components

$s \mapsto v_i(s)$ are differentiable for $1 \le i \le n$. If $\mathbf{v}, \mathbf{w}$ are differentiable vector fields along a curve c, then we have

$$\begin{aligned}
\frac{d}{ds}\langle \mathbf{v}(s), \mathbf{w}(s)\rangle &= \frac{d}{ds}\sum_{i=1}^{n} v_i(s)w_i(s) = \sum_{i=1}^{n}\frac{d}{ds}(v_i(s)w_i(s)) \\
&= \sum_{i=1}^{n}\big(v_i('s)w_i(s) + v_i(s)w_i'(s)\big) \\
&= \big\langle \mathbf{v}'(s), \mathbf{w}(s)\big\rangle + \big\langle \mathbf{v}(s), \mathbf{w}'(s)\big\rangle .
\end{aligned}$$

Thus, differentiation of the inner product of vector fields obeys a Leibniz type rule.

ii) If $\mathbf{v}\colon (a,b) \to \mathbb{R}^n$ is a differentiable vector field with $\|\mathbf{v}(s)\| = 1$, then $\mathbf{v}'$ is orthogonal to $\mathbf{v}(s)$. Hint: Apply Ex. 1 to $\langle \mathbf{v}(s), \mathbf{v}(s)\rangle = 1$.

iii) Interpret this result geometrically. Hint: Think of $\mathbf{v}$ as a curve lying on the unit sphere $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$.

Let $c\colon (a,b) \to \mathbb{R}^3$ be a regular curve, i.e., c is continuously differentiable with $c'(t) \neq 0$ for $t \in (a,b)$. **We shall assume that our curve c is parametrized by the arc-length.** That is, we assume $\|c'(s)\| = 1$ for all $s \in (a,b)$ where s is the arc-length parameter. We then say that c is of unit speed. To start with we assume that c is N-times differentiable for N sufficiently large. We shall see later that it is enough to assume that $N = 3$.

We set $\mathbf{t}(s) := c'(s)$. Then $\mathbf{t}$ is called the *unit tangent field* of c. We proceed as in the plane curves. Since $\langle \mathbf{t}(s), \mathbf{t}(s)\rangle = 1$, on differentiation with respect to s, we get $\langle \mathbf{t}'(s), \mathbf{t}\rangle + \langle \mathbf{t}, \mathbf{t}'(s)\rangle = 0$ for all $s \in (a,b)$. Hence, $\langle \mathbf{t}'(s), \mathbf{t}(s)\rangle = 0$. Therefore, $\mathbf{t}'(s)$ is orthogonal to $\mathbf{t}(s)$.

As we did in the Frenet theory of plane curves, we assume that $\mathbf{t}'(s) \neq 0$ or what is the same $c''(s) \neq 0$ for any s. Unlike the case of plane curves, there is no specific way of choosing one of the two unit vectors $\pm(c'(s)/\|c'(s)\|)$. In particular, we cannot speak of signed curvature for a space curve! However, taking the cue from the plane curves, we define $\kappa(s) := \|\mathbf{t}'(s)\| = \|c''(s)\|$. $\kappa(s)$ is called the *curvature* of c at $c(s)$. We now define the *unit normal field* $\mathbf{n}(s)$ by setting $\mathbf{n}(s) := \kappa(s)^{-1}\mathbf{t}'(s)$ so that

$$\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s). \tag{1}$$

As $\langle \mathbf{n}(s), \mathbf{n}(s)\rangle = 1$, by Ex. 1, we see that

$$\big\langle \mathbf{n}'(s), \mathbf{n}(s)\big\rangle = 0. \tag{2}$$

Hence $\mathbf{n}'(s)$ is orthogonal to $\mathbf{n}(s)$. Is it also orthogonal to $\mathbf{t}(s)$? To answer this, we differentiate the equation $\langle \mathbf{t}(s), \mathbf{n}(s)\rangle = 0$ to get $\langle \mathbf{t}'(s), \mathbf{n}(s)\rangle + \langle \mathbf{t}(s), \mathbf{n}'(s)\rangle = 0$. In view of (1) this becomes $\langle \kappa(s)\mathbf{n}(s), \mathbf{n}(s)\rangle + \langle \mathbf{t}(s), \mathbf{n}'(s)\rangle = 0$ or

$$\big\langle \mathbf{n}'(s), \mathbf{t}(s)\big\rangle = -\kappa(s). \tag{3}$$

Since we assume $\kappa(s) \neq 0$, it follows that $\mathbf{n}'(s)$ is not orthogonal to $\mathbf{t}(s)$.

Since our professed aim is to find a (positively oriented) orthonormal basis, we take the unit vector $\mathbf{b}(s)$ which is orthogonal to $\mathbf{t}(s)$ and $\mathbf{n}(s)$ in such a way that the ordered basis

$\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ is positively oriented. (More about $\mathbf{b}(s)$ in a subsection below). From (2) and (3), it follows that $\mathbf{n}'(s)$ is a linear combination of the form

$$\mathbf{n}'(s) = -\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s). \tag{4}$$

Since $\langle \mathbf{b}(s), \mathbf{b}(s)\rangle = 1$, by Ex. 1, we see that $\mathbf{b}'(s)$ is orthogonal to $\mathbf{b}(s)$ and hence lies in the span of the vectors $\mathbf{t}(s)$ and $\mathbf{n}(s)$. To find its $\mathbf{t}(s)$ and $\mathbf{n}(s)$ components, we differentiate $\langle \mathbf{t}(s), \mathbf{b}(s)\rangle = 0$ and $\langle \mathbf{n}(s), \mathbf{b}(s)\rangle = 0$. The former yields $\langle \mathbf{b}'(s), \mathbf{t}(s)\rangle + \langle \mathbf{b}(s), \mathbf{t}'(s)\rangle = 0$ which in view of (1) and the fact that $\mathbf{n}(s) \perp \mathbf{b}(s)$ becomes

$$\left\langle \mathbf{b}'(s), \mathbf{t}(s)\right\rangle = 0. \tag{5}$$

Differentiation of $\langle \mathbf{b}(s), \mathbf{n}(s)\rangle = 0$ gives

$$\begin{aligned} 0 &= \left\langle \mathbf{b}'(s), \mathbf{n}(s)\right\rangle + \left\langle \mathbf{b}(s), \mathbf{n}'(s)\right\rangle \\ &= \left\langle \mathbf{b}'(s), \mathbf{n}(s)\right\rangle + \langle \mathbf{b}(s), -\kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s)\rangle \\ &= \left\langle \mathbf{b}'(s), \mathbf{n}(s)\right\rangle + \langle \mathbf{b}(s), \tau(s)\mathbf{b}(s)\rangle \end{aligned}$$

so that we get

$$\left\langle \mathbf{b}'(s), \mathbf{n}(s)\right\rangle = -\tau(s). \tag{6}$$

From (5) and (6), it follows that

$$\mathbf{b}'(s) = -\tau(s)\mathbf{n}(s). \tag{7}$$

Putting (1), (4) and (7) we get the following Frenet-Serret formulas for a C^N-curve:

$$\begin{array}{rcccc} \mathbf{t}'(s) &=& & \kappa(s)\mathbf{n}(s) & \\ \mathbf{n}'(s) &=& -\kappa(s)\mathbf{t}(s) & & +\tau(s)\mathbf{b}(s) \\ \mathbf{b}'(s) &=& & -\tau(s)\mathbf{n}(s) & \end{array} \tag{8}$$

Now we attend to a technical matter which we had been postponing. How to get an explicit description of $\mathbf{b}(s)$ so that we shall be able to say whether or not $\mathbf{b}$ is differentiable under the given differentiability assumptions on c? In classical texts, $\mathbf{b}(s)$ is constructed as the vector or cross product of the two vectors $\mathbf{t}(s)$ and $\mathbf{n}(s)$. But there is no analogous operation on higher dimensions. Also as our approach indicates finding $\mathbf{b}(s)$ is a problem in Linear Algebra. So we proceed to achieve an explicit construction of $\mathbf{b}(s)$ in a way that will generalize to higher dimensions.

2 Vector Product on $\mathbb{R}^3$

We define a *cross-product* on a three dimensional real vector space V with an inner product: $(x, y) \mapsto \langle x, y\rangle$. We fix an orthonormal basis $\{e_i\}$ of V such that $\langle e_i, e_j\rangle = \delta_{ij}$. If you wish you may take $V = \mathbb{R}^3$ with the standard basis vectors and the Euclidean inner product $(x, y) \mapsto \langle x, y\rangle := \sum_{i=1}^3 x_i y_i$. For any *ordered* set of three points x_1, x_2, x_3 of V we define the *oriented* volume of the parallelepiped with sides Ox_i by setting:

$$\mathrm{vol}\,(x_1, x_2, x_3) = \det(\alpha_{ji}) \text{ where } x_i = \sum_j \alpha_{ji} e_j.$$

vol (x_1, x_2, x_3) is independent of the choice of the basis as above. We also have the Riesz representation theorem: For any linear map $f : V \to \mathbb{R}$ there exists a unique $u \in V$ such that $f(x) = \langle x, u\rangle$. Hint: With basis vectors e_i we take $u := \sum_i f(e_i)e_i$. We now define the *cross product* or *vector product* on V as follows:

For $x,\ y \in V$, the map $z \mapsto \mathrm{vol}\,(x, y, z)$ is linear map of V to $\mathbb{R}$ and hence by Riesz representation theorem there exists a unique vector $x \times y$ such that

$$\langle x \times y, z\rangle = \mathrm{vol}\,(x, y, z),\ \text{for all } z \in V.$$

It is easy to see that if $w := x \times y = \sum_i w_i e_i$ is the unique vector given by Riesz, then $w_j = \langle \sum_i w_i e_i, e_j\rangle = \det(x, y, e_j)$. From this we find that

$$x \times y = (x_2y_3 - x_3y_2)e_1 - (x_3y_1 - x_1y_3)e_2 + (x_1y_2 - x_2y_1)e_3.$$

This product has the following properties which are immediate consequences of well-known properties of determinants.

1. $\lambda x \times y = \lambda(x \times y) = x \times \lambda y$, for $\lambda \in \mathbb{R}$.
2. $y \times x = -x \times y$.
3. $\langle x \times y, z\rangle = \langle y \times z, x\rangle = \langle z \times x, y\rangle$.
4. $\langle x, y \times z\rangle = \langle y, z \times x\rangle = \langle z, x \times y\rangle$.

Proposition 2. *For any three vectors $x, y, z \in V$, we have*

$$x \times (y \times z) = \{(\langle x, z\rangle)y - (\langle x, y\rangle)z\}. \tag{9}$$

Proof. To show that these two vectors are equal, it is enough to show that their inner product with any vector of V (in fact, any vector in an orthonormal basis) are the same:

$$\langle v, x \times (y \times z)\rangle = \langle v, (\langle x, z\rangle)y - (\langle x, y\rangle)z\rangle\,.$$

In view of Property (4) above, it is enough to verify for an arbitrary vector v,

$$\langle v \times x, y \times z\rangle = \{\langle v, y\rangle \langle x, z\rangle - \langle x, y\rangle \langle v, z\rangle\}. \tag{10}$$

We first observe that both sides are linear in each of the variables. Hence it is enough to verify it on $\{e_i\}$. Due to symmetry we may take $y = e_1$, $z = e_2$ so that $y \times z = e_3$. Now it is easily checked that both sides of (10) are equal to $(v_1x_2 - v_2x_1)$. □

The geometric meaning of the vector or cross product $x \times y$ is that it is the vector orthogonal to x and y with the property that $\{x, y, x \times y\}$ is a basis with the same orientation as $\{e_1, e_2, e_3\}$ and is of length $\|x\|\,\|y\| \sin\theta$. This follows for example from (10). It may be noted that the latter quantity $\|x\|\,\|y\| \sin\theta$ is the area of the parallelogram spanned by x and y.

We often write $x \wedge y$ for $x \times y$.

Going back to our space curve, we find that

$$\begin{aligned}\mathbf{b}(s) &= \kappa(s)^{-1}\left[\left(c_2'(s)c_3''(s) - c_3'(s)c_2''(s)\right)e_1 - \left(c_3'(s)c_1''(s) - c_1'(s)c_3''(s)\right)e_2\right.\\ &\quad \left.+\left(c_1'(s)c_2''(s) - c_2'(s)c_1''(s)\right)e_3\right].\end{aligned}$$

Thus, if we assume that c is thrice continuously differentiable, then, $\mathbf{b}(s)$ is continuously differentiable. Thus our differentiation of $\mathbf{b}(s)$ is justified.

Theorem 3 (Frenet-Serret)**.** *Let $c : (a, b) \to \mathbb{R}^3$ be a C^3-curve parametrized by the arc-length so that $\|c'(s)\| = 1$. We further assume that $c''(s) \neq 0$. We set $\mathbf{t}(s) = c'(s)$ and $\kappa(s) := \|c''(s)\|$. We define $\mathbf{n}(s)$ by $c''(s) = \kappa(s)\mathbf{n}(s)$. Let $\mathbf{b}(s)$ be the unit vector so that $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ is a positively oriented orthonormal basis of $\mathbb{R}^3$. We have the Frenet-Serret formulas written in the matrix notation:*

$$\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}' = \begin{pmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & \tau & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{n}\\ \mathbf{b}\end{pmatrix}.$$

It should be clear that $\mathbf{t}'(s)$ measures the rate of change of the unit tangent vector field and thus is an indication of the way c is "curved." The plane spanned by $\mathbf{t}(s)$ and $\mathbf{n}(s)$ is called the *osculating plane*. It is the plane which approximates the curve to a plane curve at $c(s)$ (see Corollary 5. Its normal is $\mathbf{b}(s)$ and hence the derivative $\mathbf{b}'(s)$ which involves the τ measures how far c is from being a plane curve. τ is called the *torsion* of the curve, $\mathbf{n}(s)$ the principal normal of the curve, $\mathbf{b}(s)$ the *binormal* or the *second normal* of the curve. The plane spanned by $\mathbf{n}(s)$ and $\mathbf{b}(s)$ is called (understandably!) the *normal plane.*

3 Use of Frenet Frames

We assume that our curves are thrice continuously differentiable with $\|c'(t)\| = 1$. We however do not assume that $\kappa \neq 0$ in general.

Corollary 4. *If $c : (a, b) \to \mathbb{R}^3$ is a regular curve such that $\kappa = 0$, then c is a straight line.*

Proof. The hypothesis means that $\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s) = 0$ and hence $\mathbf{t}$ is a constant vector, say, v. Integrating $c' = v$, we get $c(s) = sv + w$ for another constant vector w. But this is the definition of the straight line passing through w with direction vector v. □

Corollary 5. *A regular curve with $\kappa(s) \neq 0$ for any s lies in a plane iff $\tau(s) = 0$ for all s.*

Proof. We use the third of the Frenet formulas. $\mathbf{b}'(s) = -\tau(s)\mathbf{n}(s) = 0$ so that $\mathbf{b}$ is a constant vector, say v. Since $\langle \mathbf{t}(s), \mathbf{b}(s)\rangle = 0$, we have

$$\begin{aligned}\frac{d}{ds}\langle c(s), \mathbf{b}(s)\rangle &= \left\langle c'(s), \mathbf{b}(s)\right\rangle + \left\langle c(s), \mathbf{b}'(s)\right\rangle\\ &= \langle \mathbf{t}(s), \mathbf{b}(s)\rangle + \left\langle c(s), \mathbf{b}'(s)\right\rangle = 0 + 0.\end{aligned}$$

Hence $\langle c(s), \mathbf{b}(s)\rangle = \alpha$, a constant. This implies that $c(s)$ lies on the plane defined by the equation $\langle c(s), \mathbf{b}(s)\rangle = \alpha$. □ □

Remark 6. It should be noted that in the above proof, we used the fact that $\kappa(s) \neq 0$ for any s (as this fact was used in the derivation of the Frenet formulas).

Corollary 7. *If all the tangent lines to a curve pass through a fixed point a, then the curve is a straight line.*

Proof. The tangent line is given by $\lambda \mapsto c(s) + \lambda c'(s)$. The hypothesis that a lies in the line means that for any s, there exists a scalar $\lambda(s) \in \mathbb{R}$ such that $a = c(s) + \lambda(s)\mathbf{t}(s)$. We differentiate this with respect to s and use first of Frenet to get:

$$0 = \mathbf{t}(s) + \lambda'(s)\mathbf{t}(s) + \lambda(s)(-\kappa(s))\mathbf{n}(s).$$

We take inner product of the above equation with $\mathbf{t}(s)$ and $\mathbf{n}(s)$. We then get

$$\lambda'(s) + 1 = 0 \qquad \text{and} \qquad \lambda(s)\kappa(s) = 0.$$

The first of these implies $\lambda(s) = -s + \alpha$ for some constant α. We then deduce from the second that $\kappa(s) = 0$ whenever $s \neq \alpha$ so that by continuity of κ, $\kappa(s) = 0$ for all s. Hence by Corollary 4, c is a straight line. □

Ex. 8. Let c be a unit speed curve lying on a sphere of radius R. Show that the curvature of c at any point is at least $1/R$.

Solution: As $\langle c, c\rangle = 1$, we see that $c \perp c'$. Hence c is the unit normal field. we differentiate $\langle t, c\rangle = 0$ to get $\langle t', c\rangle + \langle t, c'\rangle = 0$. By Frenet formula, $t' = \kappa\mathbf{n}$ and hence we deduce that $\langle \kappa N, c\rangle + \langle t, t\rangle = 0$, that is, we have $\kappa \langle N, c\rangle = -1$. Hence, we see that, using Cauchy-Schwarz inequality,

$$1 = |\langle \kappa N, c\rangle| = \kappa|\langle N, c\rangle| \leq R\kappa.$$

4 Odds and Ends

If $c\colon (a,b) \to \mathbb{R}^3$ is a non-unit speed curve, i.e., c is not parametrized by the arc-length, then the Frenet formulas cannot be used as they stand. One can find the correct formulas if one desires. However we content ourselves with the expressions for the curvature and the torsion of such curves.

Proposition 9. *Let $c\colon (a,b) \to \mathbb{R}^3$ be a regular C^3-curve, not necessarily parametrized by the arc-length. Then we have*

$$\kappa(t) = \frac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3} \tag{11}$$

$$\tau(t) = \frac{\det(c'(t), c''(t), c'''(t))}{\|c'(t) \times c''(t)\|^2}. \tag{12}$$

Proof. Let L be the length of the curve c. Let $s\colon (a,b) \to (0,L)$ be the arc-length function: $s(t) := \int_a^t \|c'(\tau)\|\, d\tau$. Then s is an increasing smooth function with an inverse $s^{-1}\colon (0,L) \to$

(a, b). If we define $\gamma := c \circ s^{-1}$, then γ is a unit speed parametrization of c. Let $\{\mathbf{t}(t), \mathbf{n}(t), \mathbf{b}(t), \kappa(t), \tau(t)\}$ denote the Frenet data at the point $c(t)$ on the curve c. Since $\gamma \circ s(t) = c(t)$, we have

$$\frac{d}{dt}\left(\gamma(s(t))\right) = \gamma'(s(t))s'(t) = c'(t) = \left\| c'(t) \right\| \mathbf{t}(t). \tag{13}$$

Hence

$$\gamma'(s(t)) = \mathbf{t}'(t) = \frac{c'(t)}{\| c'(t) \|}. \tag{14}$$

Differentiating (13), we get

$$c''(t) = \frac{d^2}{dt^2}\gamma(s(t)) = \frac{d}{dt}\big(\gamma'(s(t)) \cdot s'(t)\big) = \gamma''(s(t))(s'(t))^2 + \gamma'(s(t))s''(t). \tag{15}$$

Since γ is of unit speed, we use the Frenet equation $\gamma''(s(t)) = \kappa(t)\mathbf{n}(t)$ in (15) and obtain

$$c''(t) = (s'(t))^2\kappa(t)\mathbf{n}(t) + s''(t)\mathbf{t}(t). \tag{16}$$

As a result, we get

$$c'(t) \times c''(t) = s'(t)^3\kappa(t)\mathbf{b}(t). \tag{17}$$

Using the facts that $\mathbf{b}(t)$ is of unit norm and that $s'(t) = \| c'(t) \|$, we arrive at the following formula for the curvature of c at $c(t)$:

$$\kappa(t) = \frac{\| c'(t) \times c''(t) \|}{\| c'(t) \|^3}. \tag{18}$$

Also,

$$\mathbf{b}(t) = \frac{c'(t) \times c''(t)}{\| c'(t) \times c''(t) \|}. \tag{19}$$

We differentiate $\gamma''(s(t)) = \kappa(t)\mathbf{n}(t)$ to get

$$\begin{aligned} \gamma'''(s(t)) &= \kappa'(t)\mathbf{n}(t) + \kappa(t)\mathbf{n}'(t) \\ &= \kappa'(t)\mathbf{n}(t) + \kappa(t)\left(-\kappa(t)\mathbf{t}(t) + \tau(t)\mathbf{b}(t)\right), \end{aligned}$$

so that we obtain

$$\big\langle c'''(s(t)), \mathbf{b}(t) \big\rangle = \kappa(t)\tau(t). \tag{20}$$

On the other hand, differentiating (15), we get

$$\begin{aligned} c'''(t) = \frac{d^3}{dt^3}\gamma(s(t)) &= \gamma'''(s(t))s'(t)^3 + 3\gamma''(s(t))s'(t)s''(t) + \gamma'(s(t))s'''(t) \\ &= \gamma'''(s(t))s'(t)^3 + 3\kappa(t)s'(t)s''(t)\mathbf{n}(t) + s'''(t)\mathbf{t}(t). \end{aligned} \tag{21}$$

Taking the dot product on both sides of (21) with the vector $\mathbf{b}(t)$ (as in (19)), and using (20), we arrive at the following formula for the torsion.

$$\tau(t) = \frac{\langle c'''(t), c'(t) \times c''(t) \rangle}{\| c'(t) \times c''(t) \|^2},$$

which is what we wanted. □

Ex. 10. Find the curvature and the torsion of the following curves:
i) $\alpha(t) := (a\cos t, b\sin t, ct)$ for $t \in \mathbb{R}$ and $ab \neq 0$.
ii) $\beta(t) := (t, t^2, t^3)$, $t > 0$.
iii) $\gamma(t) := (\cosh t, \sinh t, t)$.
iv) $\delta(t) := (e^t \cos t, e^t \sin t, e^t)$.
(v) $\varepsilon(t) = (t - \cos t, \sin t, t)$.

Ex. 11. The normal plane at a point $c(t)$ of a regular curve is the plane through $c(t)$ parallel to the subspace spanned by $\{n(t), \mathbf{b}(t)\}$. Show that a curve lies on a sphere if all normal planes pass through a fixed point.

Hence, show that the curve $c(t) := (-\cos 2t, -2\cos t, \sin 2t)$, $0 \leq t \leq 2\pi$ lies on a sphere. Identify the centre and the radius of the sphere.

Ex. 12. Let $c : (0, L) \to \mathbb{R}^3$ be a unit speed curve with $\kappa > 0$ and $\tau > 0$. Define a new curve $\tilde{c}(s) := \int_0^s \mathbf{b}(\sigma) d\sigma$. Show that $\tilde{c}$ has unit speed. Let $\tilde{\kappa}$ and $\tilde{\tau}$ stand for the curvature and the torsion of $\tilde{c}$. Find the relation between the Frenet frames of c and $\tilde{c}$ and $\kappa, \tau, \tilde{\kappa}$ and $\tilde{\tau}$.

Ex. 13. i) Let $c : I \to \mathbb{R}^n$ be a differentiable curve. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Show that $(A \circ c)'(t) = A \circ c'(t)$.

ii) Let A be a rigid motion of $\mathbb{R}^3$. That is, $Ax := Tx + v$ where T is an orthogonal transformation with $\det T = 1$ and v is a fixed vector. Let $c : (a, b) \to \mathbb{R}^3$ be a curve. Let $\tilde{c} := T \circ c$ be the transformed curve. Relate the Frenet frames of c and $\tilde{c}$ and hence write down the Frenet formulas for $\tilde{c}$.

We now prove that for a regular and non-degenerate curve (i.e., $\kappa(s) \neq 0$ for all s) κ and τ determine the curve up to a congruence or a rigid motion.

Theorem 14. *Let $c, \tilde{c} \colon I \to \mathbb{R}^3$ be unit speed curves. Assume that $\kappa(s) = \tilde{\kappa}(s) > 0$ and $\tau(s) = \tilde{\tau}(s)$ for all $s \in I$. Then, if we fix $s_0 \in I$, there exists a unique rigid motion T such that $\tilde{c} = T \circ c$ on I.*

Proof. Let $s_0 \in I$ be a given fixed point. Let A be the (unique–why?) orthogonal linear transformation which takes the oriented orthonormal frame (i.e., the Frenet frame) $\{\mathbf{t}(s_0), \mathbf{n}(s_0), \mathbf{b}(s_0)\}$ at s_0 to c to the oriented orthonormal frame (i.e., the Frenet frame) $\{\tilde{\mathbf{t}}(s_0), \tilde{\mathbf{n}}(s_0), \tilde{\mathbf{b}}(s_0)\}$ at s_0 to $\tilde{c}$. Note that $\det(A) = 1$ (Why?). Let T be the rigid motion given by $Tx := Ax + v$ so that $Tc(s_0) = \tilde{c}(s_0)$. This condition means that $v := \tilde{c}(s_0) - Ac(s_0)$. Thus the rigid motion takes the point $c(s_0)$ to $\tilde{c}(s_0)$ and maps the Frenet frame of c at $c(s_0)$ to that of $\tilde{c}$ at $\tilde{c}(s_0)$. We claim that $\tilde{c} = T \circ c$. This will prove the theorem.

By Ex. 13, we see that the curve $A \circ c$ has the Frenet equations:

$$\begin{pmatrix} A \circ \mathbf{t} \\ A \circ \mathbf{n} \\ A \circ \mathbf{b} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} A \circ \mathbf{t} \\ A \circ \mathbf{n} \\ A \circ \mathbf{b} \end{pmatrix}. \tag{22}$$

(Thus $\begin{pmatrix} A \circ \mathbf{t} \\ A \circ \mathbf{n} \\ A \circ \mathbf{b} \end{pmatrix}$ and $\begin{pmatrix} \tilde{\mathbf{t}} \\ \tilde{\mathbf{n}} \\ \tilde{\mathbf{b}} \end{pmatrix}$ satisfy the same system of linear differential equations on I and at $s = s_0$ both the solutions have the same value. Hence by the uniqueness part of the

Existence and Uniqueness of a linear system of ODE, it follows that both the solutions are equal on I. That is, $\tilde{\mathbf{t}}(s) = \mathbf{t}(s)$, etc., for all $s \in I$. But $c(s) = \int_{s_0}^{s} \mathbf{t}(\sigma)d\sigma = \int_{s_0}^{s} \tilde{\mathbf{t}}(\sigma)d\sigma = \tilde{c}(s)$ by the fundamental theorem of calculus. Hence the theorem is proved.)

We may proceed as follows to show that the frames $\{A \circ \mathbf{t}, \ldots\}$ and $\{\tilde{\mathbf{t}}, \ldots\}$:

$$\begin{aligned} &\frac{d}{ds}(\left\|\tilde{\mathbf{t}} - A \circ \mathbf{t}\right\|^2 + \|\tilde{\mathbf{n}} - A \circ \mathbf{n}\|^2 + \left\|\tilde{\mathbf{b}} - A \circ \mathbf{b}\right\|^2) \\ &= 2\left(\left\langle \tilde{\mathbf{t}} - A \circ \mathbf{t}, \tilde{\mathbf{t}}' - A \circ \mathbf{t}' \right\rangle + \left\langle \tilde{\mathbf{n}} - A \circ \mathbf{n}, \tilde{\mathbf{n}}' - A \circ \mathbf{n}' \right\rangle + \left\langle \tilde{\mathbf{b}} - A \circ \mathbf{b}, \tilde{\mathbf{b}}' - A \circ \mathbf{b}' \right\rangle \right). \end{aligned}$$

The last quantity on the right of the equality sign is 0 in view of "(22)". Thus the object under question is a constant and it is 0 at $s = s_0$. □

5 Maurer-Cartan Forms and Rigidity for Space Curves

This section is optional, may not be suitable for the first reading.

Let $SO(n, \mathbb{R}) \equiv SO(n)$ be the group of all orthogonal linear maps with determinant 1. A map $A : I \subset \mathbb{R} \to SO(n, \mathbb{R})$ is said to be once continuously differentiable or C^1 if we write $A(s) := (a_{ji})$ with respect to the standard basis, the functions $s \mapsto a_{ji}(s)$ are C^1. Such an A is thought of as a curve in $SO(n)$. Since $SO(n) \subset M(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ via the map $X := (x_{ji}) \mapsto (x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, \ldots, x_{n1}, \ldots, x_{nn})$ the differentiation and integration of matrix valued functions are the same as the case of vector valued functions as indicated earlier, that is, they are carried out component-wise. In particular, for any C^1-function $B : (s, t) \to M(n, \mathbb{R})$, we have:

$$\int_s^t B(\sigma)d\sigma = (\int_s^t b_{11}(\sigma)d\sigma, \int_s^t b_{12}(\sigma)d\sigma, \ldots, \int_s^t b_{nn}(\sigma)d\sigma).$$

We then have the fundamental theorem of calculus: for any continuously differentiable function $A : I \to M(n, \mathbb{R})$ and any fixed point $s_0 \in I$,

$$A(s) = A(s_0) + \int_{s_0}^{s} A'(\sigma)d\sigma. \tag{23}$$

Ex. 15. Let $A, B : I \to M(n, \mathbb{R})$ be C^1-functions. Show that $(A \circ B)'(s) = (AB)'(s) = A'(s)B(s) + A(s)B'(s)$. Hint: If $C(s)$ is their product, then its entries are given by $c_{ij} = \sum_k a_{ik}b_{kj}$.

Example 16. The most important example for us arises out of the Frenet frames of a regular C^3-curve $c : I \to \mathbb{R}^3$ as follows. We assume as usual that $c'(s)$ and $c''(s)$ are linearly independent at each $s \in I$. We denote by $\mathbf{e}$ the standard ordered orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$. We let $\mathbf{e}(s)$ stand for the Frenet frame at the point $c(s)$. Here, of course $e_1(s) = \mathbf{t}(s), e_2(s) = \mathbf{n}(s)$ and $e_3(s) = \mathbf{b}(s)$. Then there exists an orthogonal matrix $A(s)$ such that $\mathbf{e}(s) = A(s)\mathbf{e}$ for all $s \in I$. Recall that $A(s) = (a_{ji}(s))$ where $e_i(s) = \sum a_{ji}(s)e_j$. $A \in SO(3)$ since it takes the oriented basis $\mathbf{e}$ to the orthonormal basis $\mathbf{e}(s)$ with the same orientation. From the description of the entries of A and the section on the Frenet frame and formulas, we see that $A : I \to SO(3)$ is C^1 and hence an example of the required type. Thus given a curve c we "lifted" it to a curve $A : I \to SO(3)$.

We remark that $\mathbf{e}(s)$ and $\mathbf{e}$ can be considered as 3×3-matrices if we write their components as column vectors. With this understanding the equation $\mathbf{e}(s) = A(s)\mathbf{e}$ is an equation of matrices where the RHS is the product of the two matrices.

We differentiate $\mathbf{e}(s) = A(s)\mathbf{e}$ (or equivalently, the equation $e_i(s) = A(s)e_i$) to get

$$\mathbf{e}'(s) = A'(s)\mathbf{e} = A'(s)A^{-1}(s)\mathbf{e}(s).$$

Let $C(s) := A'(s)A(S)^{-1}$. In this notation the Frenet equation (8) can be recast as

$$\mathbf{e}'(s) = C(s)\mathbf{e}(s) \qquad \text{for } s \in I. \tag{24}$$

$C(s)$ is called the Cartan (or Maurer-Cartan) matrix of A or the curve c.

Motivated by this, even in the general case of any such A, we define the Maurer-Cartan form or the *Cartan matrix* of A to be the matrix $C_A(s) := A'(s)A(s)^{-1} = A'(s)A^{-1}(s)$. When there is no possible source of confusion, we write $C(s)$ in place of $C_A(s)$.

Lemma 17. *The Cartan matrix C of any C^1-curve A in $SO(n)$ is skew-symmetric.*

Proof. Since $A(s)$ is orthogonal, $A^{-1} = A^*$. Hence we differentiate $A(s)A^*(s) = I$ to get $A'(s)A^*(s) + A(s)A'(s)^* = 0$. Or, $A'(s)A^*(s) + (A'(s)A^*(s))^* = 0$ and hence the lemma. □

Now the rigidity result Theorem 14 can be reformulated and reproved in a more elegant way as follows:

Theorem 18. *Let $c_i : I \to \mathbb{R}^3$ be regular C^3-curves parametrized by their arc length. Assume that their Cartan matrices C_i are equal at all points $s \in I$. Then there exists a rigid motion of $\mathbb{R}^3$ taking c_1 to c_2.*

Proof. First of all note that our hypothesis is same as the equality of the functions: $\kappa_1(s) = \kappa_2(s)$, $\tau_1(s) = \tau_2(s)$ for all s and this is same as the hypothesis of Theorem 14. We use the obvious notation below.

We can write $A_1(s) = A_2(s)B(s)$ for some (necessarily orthogonal matrix) $B(s)$. Since the Cartan matrices of A_i are equal, we must have

$$\begin{aligned}
C_1(s) &= C_{A_2(s)B(s)}(s) \\
&= (A_2(s)B(s))' \, (A_2(s)B(s))^{-1} \\
&= \left(A_2'(s)B(s) + A_2(s)B'(s)\right)\left(B^{-1}(s)A_2^{-1}(s)\right) \\
&= A_2'(s)A_2^{-1}(s) + A_2(s)B'(s)B^{-1}(s)A_2^{-1}(s) \\
&= C_2(s) + A_2(s)B'(s)B^{-1}(s)A_2^{-1}(s).
\end{aligned}$$

Hence we deduce that $A_2(s)B'(s)B^{-1}(s)A_2^{-1}(s) = 0$ or $B'(s)B^{-1}(s) = 0$ and so $B'(s) = 0$. That is, $B(s) = B$, a constant (orthogonal matrix).

The up-shot of the last paragraph is that the Frenet equations, (24) of the curves imply that the curves c_1 and $B \circ c_2$ have the same tangent vectors at all points s.

If c is any C^1 curve, then we have from the fundamental theorem of calculus, that

$$c(s) = c(s_0) + \int_{s_0}^{s} c'(\sigma) d\sigma.$$

Applying this to the curve $B \circ c_2$, where s_0 is an arbitrary point in I, we get

$$\begin{aligned} B \circ c_2(s) &= B \circ c_2(s_0) + \int_{s_0}^{s} B \circ c_2(\sigma) d\sigma \\ &= B \circ c_2(s_0) - c_1(s_0) + c_1(s_0) + \int_{s_0}^{s} c_1'(\sigma) d\sigma \\ &= B \circ c_2(s_0) - c_1(s_0) + c_1(s). \end{aligned}$$

Thus the curves $T \circ c_2$ and c_1 differ by the translation by $B \circ c_2(s_0) - c_1(s_0)$. □

6 Existence of Space Curves with the given Curvature and Torsion

We shall now see how this matrix formulation helps us to prove the Fundamental theorem on the existence and uniqueness of space curves with the given curvature and torsion. Given two functions $\kappa, \tau : I \to \mathbb{R}$, the problem is to find a C^3-curve $c : I \to \mathbb{R}^3$ such that its curvature and torsion are κ and τ. Given these function, we consider the associated Cartan matrix $C(s) := \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}$. We then look for the solution of the matrix differential equation $A'(s)A(s)^* = C(s)$ with the initial condition $A(s_0) = I$. The matrix equation can be written as $A'(s)A(s)^{-1} = C(s)$ or $A'(s) = C(s)A(s)$.

7 General Existence Theorem of ODE

However the above approach will not be useful if we want to prove such an existence result for space curves. Here what we do is typical of many theorems in analysis. We start with an approximate "solution" of the given equation and then *iterate* it to get better and better approximate solutions which converge (fortunately!) to the desired solution. We thus investigate the existence of the solution of the matrix differential equation $A'(s) = C(s)A(s)$ with the initial condition $A(s_0) = I$ where $C : I \to M(k, \mathbb{R})$, the $(k \times k)$-matrices. We have the following fundamental

Theorem 19. *Let $C : I \to M(k, \mathbb{R})$ a continuous map be given on a closed and bounded interval I. Let $s_0 \in I$ be a fixed point. Then there exists a unique $A : I \to M(k, \mathbb{R})$ such that $A'(s) = C(s)A(s)$ with the initial value $A(s_0) = \mathrm{Id}$.*

Furthermore, if $C(s)$ is skew-symmetric for all $s \in I$, then $A(s) \in O(k)$ for all s.

Proof. In fact, we rather look into the equivalent problem of solving the integral equation

$$A(s) = \mathrm{Id} + \int_{s_0}^{s} C(\sigma)A(\sigma) d\sigma.$$

(The problems are equivalent by the fundamental theorem of calculus (23).)

We need some preliminary facts. For $A \in M(n, \mathbb{R})$, we let $\|A\| := \max_{1 \le i,j \le k}\{|a_{ij}|\}$. Then it is easy to check that $\|AB\| \le k\,\|A\|\,\|B\|$. In particular, $\|A^2\| \le k\,\|A\|^2$. Since C is continuous, there exists a constant M such that $\|C(s)\| \le M$ for all $s \in I$. We also invite the reader to solve

Ex. 20. Let $A, A_n \in M(k, \mathbb{R})$ for $n = 1, 2, \ldots$. Then $\|A - A_n\| \to 0$ iff the matrix entries $a_{ij}^n \to a_{ij}$ as $n \to \infty$ for all $1 \le i, j \le k$.

(*We should remark that this is not the only possible "norm" on* $M(n, \mathbb{R})$. *In case, the reader knows about the operator norm, and if he uses it in place of the norm above, then there will be no factor of* k *in the inequality* $\|AB\| \le \|A\|\,\|B\|$.)

To prove the existence part of the theorem, it is enough to show that for any $s_1, s_2 \in I$ with $s_1 < s_0 < s_2$ the approximate solutions A_n converge uniformly to a solution of the integral equation. Then the existence on I follows.

To attend to the details we let $A_0(s) = \mathrm{Id}$ and feed it into the integral equation $A(s) = \mathrm{Id} + \int_{s_0}^{s} C(\sigma)A(\sigma)d\sigma$. That is, we define

$$\begin{aligned}
A_1(s) &= \mathrm{Id} + \int_{s_0}^{s} C(\sigma)A_0(\sigma)d\sigma \\
A_2(s) &= \mathrm{Id} + \int_{s_0}^{s} C(\sigma)A_1(\sigma)d\sigma \\
\vdots &\ \vdots\ \vdots \\
A_k(s) &= \mathrm{Id} + \int_{s_0}^{s} C(\sigma)A_{k-1}(\sigma)d\sigma.
\end{aligned}$$

One easily shows by induction that

$$\|A_{n+1}(s) - A_n(s)\| \le k^{n-1}M^n \frac{|s - s_0|}{n!} \le k^{n-1}M^n \frac{|s_2 - s_1|}{n!}$$

The term on the RHS of the above estimate is the n-th term of the Taylor expansion of $(1/k)e^{kM|s_1 - s_2|}$ which goes to 0 as $n \to \infty$. Thus the approximate solutions $A_n(s)$ converge uniformly on $[s_1, s_2]$ to a continuous solution $A(s)$ of the integral equation. Since the right side of the integral equation is an indefinite integral of the continuous function A, the left side A is indeed differentiable. It is also a solution of the given matrix differential equation.

If A and B are two solutions of the ODE $A' = CA$ with the IC (initial condition) $A(s_0) = \mathrm{Id}$, then by repeating the above argument, we show that $\|A_n - B_n\| \to 0$ as $n \to \infty$.

Proof of the orthogonality of A is similar to that of Lemma 17: Differentiate A^*A to get $(A')^*A + A^*A' = (CA)^*A + A^*CA = A^*(-C)A + A^*CA = 0$. Hence A^*A is a constant which is the identity at $s = s_0$. Therefore A is orthogonal. If C is skew-symmetric, uniqueness can also be proved by adopting the argument in the proof of Theorem 18. □

From this it is easy to deduce the following existence and uniqueness theorem for space curves:

Theorem 21. *Let $\kappa, \tau : I \to \mathbb{R}$ be continuous. Assume that $\kappa(s) \neq 0$ for $s \in I$. Let $x_0 \in \mathbb{R}^3$ be given. Let $\{e_1(s_0), e_2(s_0), e_3(s_0)\}$ be an (ordered) orthonormal basis having the same orientation as that of the standard basis. Then there exists a unique unit speed C^3-curve $c : I \to \mathbb{R}^3$ such that its curvature and torsion are the given functions κ and τ.*

Proof. Take $n = 3$ in the above theorem. Let $\kappa : I \to \mathbb{R}$ and $\tau : I \to \mathbb{R}$ be given. Let $C(s)$ be the Cartan matrix defined at the beginning of this section. We fix a point $x_0 \in R^3$ and take the orthonormal frame at x_0 to be the standard frame. We then take $e_i(s) := A(s)e_i$ as the Frenet frames. In particular, $e_1(s)$ as tangent vectors so that the curve is given by

$$c(s) := \int_{s_0}^{s} e_1(\sigma) d\sigma.$$

It is now an easy exercise to show that c is a curve with the given curvature and torsion. But however the Frenet frame of c at $c(s_0)$ may not be $\{e_1(s_0), e_2(s_0), e_3(s_0)\}$. To remedy this, we consider $\tilde{c} := B \circ c$, where B is the orthogonal matrix (with determinant 1) which takes the standard basis to this basis. Then $\tilde{c}$ is the required one. □

8 The Case of Plane Curves

This section is more demanding than the earlier; however this will motivate the study of matrix differential equations and exponential of matrices etc. The serious students are very strongly urged to go through this section.

It is easier to solve the matrix equation $A' = CA$ in a more concrete way in the case of a plane curve and hence the analogue of Theorem 21 for plane curves. So assume that we are given a continuous function $k : I \to \mathbb{R}$ on an interval I. We wish to find a curve $c : I \to \mathbb{R}^2$ such that its curvature $\kappa_c(s) = k(s)$ on I. Now the corresponding Cartan matrix is given by $C(s) = \begin{pmatrix} 0 & k(s) \\ -k(s) & 0 \end{pmatrix} = k(s) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = k(s)J$. Thus the matrix differential equation we wish to solve becomes $A'(s)A(s)^{-1} = k(s)J$. The whole thing smacks of "exponential" and hence we may try

$$A(s) := e^{J \int_{s_0}^{s} k(\sigma)d\sigma} = \begin{pmatrix} \cos \int_{s_0}^{s} k(\sigma)d\sigma & \sin \int_{s_0}^{s} k(\sigma)d\sigma \\ -\sin \int_{s_0}^{s} k(\sigma)d\sigma & \cos \int_{s_0}^{s} k(\sigma)d\sigma \end{pmatrix}.$$

More leisurely, we define $B(s) := \int_{s_0}^{s} C(\sigma)d\sigma = \left(\int_{s_0}^{s} k(\sigma)d\sigma \right) J$. Then $C(s)$ is the derivative of $B(s)$ and they commute with each other (only in this case and hence needs verification!). Therefore $(B^n)' = nB^{n-1}B' = nB^{n-1}C = nCB^{n-1}$ so that $(e^B)' = Ce^B$. Here the exponential of a matrix is defined by the infinite series $\exp(A) \equiv e^A := \sum_{n=0}^{\infty}(A^n/n!)$. For more on the exponential, see the appendix. Thus if we take $A(s) := e^{J \int_{s_0}^{s} k(\sigma)d\sigma}$, then we have solved the matrix differential equation $A'(s) = C(s)A(s)$.

Now to get the curve c from this easy: For, what the foregoing tells us is that the Frenet frame is uniquely determined if we fix any orthonormal frame $\{e_1(s_0), e_2(s_0)\}$ with the same orientation as the standard basis. The Frenet frame at $c(s)$ is given by $\{e_1(s), e_2(s)\} = A(s)\{e_1(s_0), e_2(s_0)\}$. In particular, if we fix an initial point $(x_0, y_0) \in \mathbb{R}^2$ as $c(s_0)$, then the tangent field is given by $e_1(s) = A(s)e_1(s_0)$ and hence the curve is given by $c(s) := c(s_0) + \int_{s_0}^{s} e_1(\sigma)d\sigma$. That is, the curve is given by

$$c(s) = (x_0 + \int_{s_0}^{s} (\cos \int_{s_0}^{s} k(\sigma)d\sigma), y_0 + \int_{s_0}^{s} (-\sin \int_{s_0}^{s} k(\sigma)d\sigma)).$$

Ex. 22. It is an instructive exercise to derive the expression for the plane curve by completely elementary means without the use of the matrix exponential.

Exercise: The Exponential Map in $M(n, \mathbb{R})$

The following set of exercises introduces the exponential map in $M(n, \mathbb{R})$ and its properties:

1) Show that if $f: U \subset \mathbf{E} \to \mathcal{F}$ is differentiable at x, then it remains so if $\mathbf{E}$ and F are endowed with equivalent norms. What is $f'(x)$ in this case?

2) For $X \in M(n, \mathbb{R})$, $X := (x_{ij})$, let

$$\| X \| := \max_{1 \le i,j \le n} |x_{ij}|$$

be the max norm. It is equivalent to the operator norm on elements of $M(n, \mathbb{R})$ *viewed as linear operators on* $\mathbb{R}^n$.

3) We have $\| AB \| \le n \| A \| \| B \|$ for all $A, B \in M(n, \mathbb{R})$ and $\left\| A^k \right\| \le n^{k-1} \| A \|^k$.

4) A sequence $A_k \to A$ in the max norm iff $a_{ij}^k \to a_{ij}$ for all $1 \le i, j \le n$ as $k \to \infty$. Here we have $A_k := (a_{ij}^k)$ etc.

5) If $\sum_{k=0}^{\infty} \| A_k \|$ is convergent, then $\sum_0^{\infty} A_k$ is convergent to an element A of $M(n, \mathbb{R})$.

6) For any $X \in M(n, \mathbb{R})$, the series $\sum_0^{\infty} \frac{X^k}{k!}$ is convergent. We denote the sum by $\exp(X)$ or by e^X.

7) For a fixed $X \in M(n, \mathbb{R})$ the function $f(t) := e^{tX}$ satisfies the matrix differential equation $f'(t) = Xf(t)$, with the initial value $f(0) = I$. Hint: The (i, j)-th entry of $f(t)$ is a power series in t and use 4).

8) Set $g(t) := e^{tX} e^{-tX}$ and conclude that e^{tX} is invertible for all $t \in \mathbb{R}$ and for all $X \in M(n, \mathbb{R})$.

9) There exists a unique solution for $f'(t) = Af(t)$ with initial value $f(0) = B$ given by $f(t) = e^{tA}B$. Hint: If g is any solution consider $h(t) = g(t)\, e^{-tA}$.

10) Let A, $B \in M(n, \mathbb{R})$. If $AB = BA$ then we have $e^{A+B} = e^A e^B = e^B e^A = e^{B+A}$. Hint: Consider $\phi(t) := e^{t(A+B)} - e^{tA} e^{tB}$.

11) For A, $X \in M(n, \mathbb{R})$ we have $e^{AXA^{-1}} = Ae^X A^{-1}$.

An apology! This article was written several years ago and I have not gone through it recently. I would appreciate if the readers send me corrections and suggestions for improvement.

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