Cauchy's Theorem $\implies$ Fundamental Theorem of Algebra

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Ask anybody for a proof of the fundamental theorem of algebra. Invariably, you will get the answer that it is an easy corollary of Liouville's theorem of complex analysis. The aim of this article is to give three proofs all of which use only Cauchy's theorem from complex analysis.

Proof 1. The case when the polynomial is of degree 1 is trivial. So we assume that the polynomial is of degree at least 2, say, $P(z) := \sum_{k=0}^{n} a_k z^k$.

Assume that $P(z)$ be a polynomial of degree at least 2 with *real* coefficients. Assume that $P(z) \neq 0$ for $z \in \mathbb{C}$. Let $R > 0$. Let γ_R denote the piecewise smooth path consisting of the line segment from $-R$ to R and the semicircle from R to $-R$ with centre at 0. We assume that pieces of this path are parametrized the standard way. Since $P(z)$ never vanishes, the function $f(z) := 1/P(z)$ is holomorphic on $\mathbb{C}$. By Cauchy's theorem, we have $\int_{\gamma_R} f(z)\,dz = 0$.

Recall the standard estimate for polynomials: there exists $M > 0$ such that

$$|P(z)| \geq \frac{1}{2}|a_n||z|^n \text{ for } |z| \geq M. \tag{1}$$

Let $C_R(t) := Re^{it}$ for $0 \leq t \leq \pi$ be the semicircle part and $L_R(t) = t$ for $-R \leq t \leq R$ be the line segment part of the path γ_R. It follows from (1) that $|\int_{C_R} |f(z)\,dz$ can be made arbitrarily small by choosing R large enough. The details follow.

Let $\varepsilon > 0$ be given. Choose $R > \max\left\{M, \left(\frac{2\pi}{|a_n|\varepsilon}\right)^{1/(n-1)}\right\}$. Then, using (1), $|1/f(z)| < \frac{2}{|a_n|}\frac{1}{R^n}$. By the ML inequality for the path integrals, we have

$$|\int_{C_R} f(z)\,dz| < \pi R \frac{2}{|a_n|}\frac{1/R^n}{=}\frac{2\pi}{|a_n|}\frac{1}{R^{n-1}} < \varepsilon.$$

Hence it follows that $0 = \lim_{R\to\infty} \int_{\gamma_R} f(z)\,dz = \int_{\mathbb{R}} f(t)\,dt$. Since f is never zero on $\mathbb{R}$, it follows that $f(t) > 0$ for all t or $f(t) < 0$ for all $t \in \mathbb{R}$. Hence $\int_R f(t)\,dt > 0$ or < 0. This is a contradiction. $\square$

Question: Where are the second and third proofs?