The Fundamental Theorem of Algebra: Proofs via Differential Topology

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We use very elementary concepts from differential topology to prove the result.

Theorem 1 (Fundamental Theorem of Algebra)**.** *If P is a complex polynomial of degree $n \geq 1$ then there is root of P in $\mathbb{C}$.*

Proof. The map $P\colon \mathbb{C} \to \mathbb{C}$ can be thought of a map $P\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $P(x, y) = (u(x, y), v(x, y))$ where $P(x + iy) = u(x + iy) + iv(x + iy)$. Now $z = x + iy$ is a critical point of P iff $P'(z) = 0$ and hence there are only finitely many critical values of P. Let F be the finite set of critical values of P. Since $P(z) \to \infty$ as $z \to \infty$, we can extend P as map of the one-point compactification S^2 of $\mathbb{R}^2$.

For any $\alpha \in \mathbb{C}$, the set $P^{-1}(\alpha)$ consists at most n points. Let $d = d(\alpha)$ be the number of points in $P^{-1}(\alpha)$. If α is a regular value and $P^{-1}(\alpha) = \{z_1, \dots, z_k\}$, then for each i, there exists an open neighbourhood U_i of z_i which is mapped diffeomorphically onto an open set V_i in $\mathbb{C}$. Without loss of generality, we may assume that V_i's are pair-wise disjoint. Let V be an open connected neighbourhood of α in $\cap V_i$. Then $W_i := U_i \cap P^{-1}(V)$ is mapped diffeomorphically onto V. Note that $P^{-1}(V)$ is the disjoint union of W_i's. It follows that $d(\alpha)$ is locally constant on $\mathbb{C} \setminus F$. Since $\mathbb{C} \setminus P^{-1}(F)$ is connected, this must be a constant d on $\mathbb{C} \setminus P^{-1}(F)$. This constant d cannot be zero, since otherwise, P maps $\mathbb{C}$ into F and hence is a constant. This shows that the image of P contains $(\mathbb{C} \setminus F) \cup F = \mathbb{C}$. In particular, the value 0 is assumed by P. $\square$

Ex. 2. This outlines another proof as a series of steps.

(a) Let h_N and h_S denote the stereographic projections of S^2 onto $\mathbb{C}$ from the north and south poles respectively. Define $f\colon S^2 \to S^2$ by setting

$$f(z) = \begin{cases} (h_N^{-1} \circ P \circ h_N)(z), & z \in \mathbb{C} \\ N & z = N \end{cases}$$

Show that f is smooth. *Hint:* The smoothness at N follows from observing that

$$(h_S \circ f \circ h_S^{-1})(z) = \frac{z^n}{\overline{a}_0 z^n + \overline{a}_1 z^{n-1} + \cdots + \overline{a}_n}.$$

(b) Assume that $P(z) \neq 0$ for $z \in \mathbb{C}$. Then the image $K = f(S^2)$ is compact, $S \notin K$ so that S has an open neighbourhood disjoint from K.

(c) If $P'(z) \neq 0$, show that $f(h_N^{-1}(z))$ lies in the interior of K.

(d) Show that the boundary of K consists of infinitely many points. *Hint:* Look at the southernmost point of intersection of Y with any great circle starting from N.

(e) Conclude that P' has an infinite number of zeros.