Arzela-Ascoli Theorem and Applications

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The theorem of the title gives an immensely useful criterion of compactness of subsets of $C(X)$ where X is a compact metric space and $C(X)$ is given the sup norm metric.

Definition 1. Let X and Y be metric spaces. A family $\mathcal{A}$ of functions from X to Y is said to be *equicontinuous* on X if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that

I may need to give more details here.

$$d(x_1, x_2) < \delta \Rightarrow d(f(x_1), f(x_2)) < \varepsilon \qquad \text{for all } f \in \mathcal{A}.$$

Ex. 2. Any member of an equicontinuous family is uniformly continuous.

Ex. 3. Let $f\colon X \to Y$ be any continuous function. Then $\mathcal{A} := \{f\}$ is equicontinuous iff ....

The following two exercises give two of the most important ways equicontinuous families arise.

Ex. 4. Let X be a compact metric space. Let $F\colon X \times X \to Z$ be continuous. Let $f_y(x) := F(x, y)$. Then $\mathcal{A} := \{f_y : y \in X\}$ is equicontinuous.

Ex. 5. Let $X \subset \mathbb{R}^n$ be convex and open. Let $\mathcal{A}$ be a family of differentiable functions from X to $\mathbb{R}^m$. Assume that there exists $M > 0$ such that $\|Df(x)\| \leq M$ for all $x \in X$. Then $\mathcal{A}$ is equicontinuous.

Theorem 6 (Arzela-Ascoli Theorem)**.** *Let X be a compact metric space. Let $C(X, \mathbb{K})$ be given the sup norm metric. ($\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$.) Then a set $\mathcal{B} \subset C(X)$ is compact iff $\mathcal{B}$ is bounded, closed and equicontinuous.*

Proof. Assume that $\mathcal{B}$ is compact. Then $\mathcal{B}$ is closed and totally bounded since $C(X)$ is a complete metric space. Given $\varepsilon > 0$ there exists $f_i \in \mathcal{B}$ for $1 \leq i \leq n$ such that $\mathcal{B} \subset \cup_i B(f_i, \varepsilon)$. Let δ_i be chosen by the uniform continuity of f_i for the given ε. Let δ be the minimum of the δ_i's. This δ does the job.

Now assume that $\mathcal{B}$ is bounded, closed and equicontinuous. We show that $\mathcal{B}$ is totally bounded. Let $\varepsilon > 0$ be given. Let M be such that $|f(x)| \leq M$ for all $x \in X$ and $f \in \mathcal{B}$. Using the equicontinuity of $\mathcal{B}$ we get a δ. We can find $x_i \in X$, $1 \leq i \leq m$ such that $X = \cup_{i=1}^m B(x_i, \delta)$. Since $B[0, M]$ the closed ball of radius M centred at 0 in $\mathbb{K}$ is compact and hence totally bounded we can find $y_j \in B[0, M]$, $1 \le j \le n$ such that $B[0, M] \subset \cup B(y_j, \varepsilon)$. Let $A := \{\alpha \colon \{x_i\} \to \{y_j\}\}$. Then $|A| = n^m$. For $\alpha \in A$ let

$$U_\alpha := \{f \in \mathcal{B} : |f(x_i) - \alpha(i)| \le \varepsilon\}.$$

Then the diameter of U_α is at most 4ε and their union cover $\mathcal{B}$. (Draw a picture!) □

Ex. 7. Let X be a compact metric space. Let $C(X, \mathbb{R}^n)$ be given the metric

$$d(f, g) := \sup_{x \in X} \{\| f(x) - g(x) \|\}.$$

Prove an analog of Arzela-Ascoli theorem for subsets of this space.

Ex. 8. Let X and Y be compact metric spaces. Then $\mathcal{B} \subset C(X, Y)$ is compact iff it is bounded and equicontinuous.

Ex. 9. Application to Complex Analysis. Let $U \subset \mathbb{C}$ be open. Let $\{f_n\}$ be a sequence of holomorphic functions on U. Assume that for each compact subset K of U there is a constant M_K such that $|f_n(z)| \le M_K$ for $z \in K$ and $n \in \mathbb{N}$. Then there is a subsequence which converges uniformly to a holomorphic function on compact subsets of U. *Hint:* Use Cauchy integral formula to obtain equicontinuity.

This result is used in proving Riemann Mapping theorem in Complex Analysis.

Ex. 10. Application to Functional Analysis. Let $K \in C([0, 1] \times [0, 1])$. For $f \in C[0, 1]$ define the integral operator

$$T_K f(x) := \int_0^1 K(x, y) f(y)\, dy.$$

Then $T_K \colon (C[0, 1], \|\cdot\|_\infty) \to (C[0, 1], \|\cdot\|_\infty)$ is linear and *compact* in the sense that $\{T_K f : \| f \| \le 1\}$ has compact closure.

Theorem 11. Application to ODE — Peano's Theorem. *Let f be a continuous function from a neighbourhood U of $(0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$. Then there exists an $\varepsilon > 0$ such that the initial value problem*

$$\frac{dx}{dt} = f(t, x(t)), \qquad x(0) = x_0$$

has a solution x on $[0, \varepsilon]$.

Proof. Without loss of generality assume that U is of the form $[-\delta, \delta] \times B[x_0, r]$. Let M be a bound for f on U. Let $\varepsilon := \min\{\delta, r/M\}$. Define a sequence (x_n) on $[0, \varepsilon]$ as follows:

$$x_n(t) := \begin{cases} x_0 & t \in [0, \varepsilon/n], \\ x_0 + \int_0^t -\varepsilon/n f(s, x_n(s))\, ds & t \in (\varepsilon/n, \varepsilon]. \end{cases}$$

Observe that these formulas determine the function x_n on $[0, \varepsilon]$ since its values on $(k\varepsilon/n, (k+1)\varepsilon/n]$ are determined by its values on $[0, k\varepsilon/n]$ for $1 \le k \le n-1$ and its values on $[0, \varepsilon/n]$ are given. The family $\{x_n\}$ is equicontinuous on $[0, \varepsilon]$. Let (x_{n_k}) converge to x. Then x satisfies an integral equation which is equivalent to the given DE on $[0, \varepsilon]$. □

The next problem is to show you how Arzela-Ascoli is exploited in seemingly unlikely contexts. We introduce some notation. Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ be defined as follows:

$$\varphi(x) := \begin{cases} e^{-\frac{1}{1-\|x\|^2}} & \text{for } \|x\| \leq 1 \\ 0 & \text{for } \|x\| \geq 1. \end{cases}$$

For any $\varepsilon > 0$, we let $\varphi_\varepsilon(x) := \varepsilon^{-n}\varphi(x/\varepsilon)$ for $x \in \mathbb{R}^n$. Recall the convolution: $f * g(x) := \int_{\mathbb{R}^n} f(x-y)g(y)\,dy$, whenever the integral makes sense. (For more details, see the section on Approximate Identities.)

Ex. 12. Let Ω be a bounded open subset of $\mathbb{R}^n$. Let $\mathcal{A}$ be a subset of $L^p(\Omega)$. Then $\mathcal{A}$ is relatively compact iff $\mathcal{A}$ has the following properties: i) $\|\varphi_\varepsilon * f - f\|_p \to 0$ uniformly in $f \in \mathcal{A}$ as $\varepsilon \to 0$ and ii) $\|f\| \leq M$ for some M for all $f \in \mathcal{A}$. *Hint:* Enough to show that $\mathcal{A}$ is totally bounded. Let δ be given. Choose $\varepsilon > 0$ such that $\|\varphi_\varepsilon * f - f\| < \delta/2$. Enough to find a $\delta/2$-net for $\mathcal{B} := \{\varphi_\varepsilon * f : f \in \mathcal{A}\}$. Show that $\mathcal{B}$ is uniformly bounded and equicontinuous.