The Fundamental Theorem of Algebra

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Abstract

The aim of this article is to give a variety of simple and elementary proofs of the fundamental theorem of algebra.

1 A Fundamental Lemma

Throughout our discussion, we let $P(z) := \sum_{k=0}^n a_k z^k$ be a (nonconstant) polynomial with coefficients $a_k \in \mathbb{C}$, $n \geq 1$ and $a_n \neq 0$. The following lemma is fundamental for all the proofs. It says that the nonconstant polynomial functions are proper maps.

Lemma 1. *There exists $N > 0$ such that $|P(z)| > \frac{1}{2}|a_n| |z|^n$ for $|z| > N$. In particular, $|P(z)| \to \infty$ as $|z| \to \infty$.*

Proof. We have

$$\left|\frac{P(z)}{z^n}\right| = \left|a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| \geq |a_n| - \left|\sum_{j=0}^{n-1} \frac{a_j}{z^{n-j}}\right| \tag{1}$$

If $|z| \geq 1$, then $|z|^r \geq |z|$ for $r \geq 1$ so that $\frac{1}{|z^{n-j}|} \leq \frac{1}{|z|}$ or $-\frac{1}{|z^{n-j}|} \geq -\frac{1}{|z|}$ since $n - j \geq 1$. In view of this, (1) yields

$$\left|\frac{P(z)}{z^n}\right| \geq |a_n| - \sum_{j=0}^{n-1} \frac{|a_j|}{|z|}, \qquad \text{for } |z| \geq 1. \tag{2}$$

If we set $M := \sum_{j=0}^{n-1} |a_j|$, we get from (2),

$$\left|\frac{P(z)}{z^n}\right| \geq |a_n| - \frac{M}{|z|}, \qquad \text{for } |z| \geq 1. \tag{3}$$

If we choose z so that $|a_n| - \frac{M}{|z|} > |a_n|/2$, i.e. if $N > \frac{2M}{|a_n|}$ and $|z| > N$, we get the result from (3). $\square$

2 An Elementary Proof

This proof uses the fact that a real valued continuous function on a compact space attains its minimum and some basic facts on the complex e exponential function. While the proof may be considered as elementary, the reader is encouraged to go through the technical remarks at the end of this section. We outline the proof to highlight the ideas involved.

Since $|P(z)| \to \infty$ as $|z| \to \infty$, we can find $R > 0$ such that $|P(z)| < |P(0)|$ for all $|z| \geq R$. The continuous real valued function $|P(z)|$ attains a minimum, say at z_0, on the compact set $|z| \leq R$. By translation we may assume that $z_0 = 0$. By multiplying by a constant, we may assume that $a_0 = P(0)$ is real and nonnegative. Let $m \geq 1$ be chosen so that $P(z) = a_0 - \sum_{k=0}^{n} a_k z^k$ with $a_m \neq 0$. Choose $\omega \in \mathbb{C}$ so that $b = a_m \omega^m$ is real and positive. Then for all r sufficiently small and positive, we have

$$\operatorname{Re} P(r\omega) = a_0 - br^m + O(r^{m+1}) \text{ and } \operatorname{Im} P(r\omega) = O(r^{m+1}).$$

So, unless $a_0 = 0$, we have

$$|P(r\omega)| = a_0 - br^{m+1} + O(r^{m+1}),$$

and $z_0 = 0$ is not a point of minimum for $|P|$. Hence, $a_0 = P(0) = 0$. We now work out the details of this outline.

Lemma 2. *$|p|$ has a minimum in $\mathbb{C}$. That is, there exists $z_0 \in \mathbb{C}$ such that $|P(z_0)| \leq |P(z)|$ for $z \in \mathbb{C}$.*

Proof. Since $B[0, k]$ is compact for any $k \in \mathbb{N}$, the continuous function $|p|$ attains its minimum m_k at some point, say, $z_k \in B[0, k]$. From Lemma 1 it follows that there exists $N \in \mathbb{N}$ such that $|P(z)| > m_1$ if $|z| > N$. We claim that $m := m_N$ is the minimum of $|p|$ in $\mathbb{C}$. For, if $|z| \leq N$, then $|P(z)| \geq m_N$ whereas, if $|z| > N$, then $|P(z)| \geq m_1 = |P(z_1)| \geq m_N$, since $z_1 \in B[0, N]$. $\square$

Theorem 3. *Every nonconstant polynomial with complex coefficients has a root in $\mathbb{C}$.*

Proof. We shall give two versions of a proof. I prefer the first proof as it does not try to hide the basic idea behind a sleek choice of notation.

First Version

We write $p(z) = p(z - z_0 + z_0) = \sum_{k=0}^{n} a_k[(z - z_0) + z_0]^k$. Using binomial theorem, we can rewrite this as $p(z) = \sum_{k=0}^{n} b_k(z - z_0)^k$ for some $b_k \in \mathbb{C}$. By hypothesis, $|b_0| = |p(z_0)| \leq |p(z)|$ for $z \in \mathbb{C}$. If $|b_0| \neq 0$, we choose $z = z_0 + re^{it}$ in a neighbourhood of z_0 so carefully that $|p(z)| < |b_0|$.

Let $k \geq 1$ be the first integer such that $b_k \neq 0$. We have

$$|p(z)| = \left|\sum_{j=0}^{n} b_j r^j e^{ijt}\right| \leq \left|b_0 + b_k r^k e^{ikt}\right| + \left|\sum_{j \geq k+1} b_j r^j e^{ijt}\right|. \tag{4}$$

We now choose $t \in \mathbb{R}$ so that $r^k b_k e^{it}$ is a negative multiple of b_0. We fix such a t. We then have $\left|b_0 + r^k b_k e^{it}\right| = |b_0| - r^k |b_k|$. From (4) we deduce that

$$|p(z)| \leq |b_0| - r^k |b_k| + \left| \sum_{j \geq k+1} b_j r^j e^{ijt} \right|. \tag{5}$$

The function $r \mapsto \sum_{j \geq k+1} b_j r^j e^{ijt}$ is continuous and is 0 at $r = 0$. Hence for $\varepsilon = \frac{1}{2} r^k |b_k|$, there is a $\delta > 0$ such that

$$\left| \sum_{j \geq k+1} b_j r^j e^{ijt} \right| < \frac{1}{2} r^k |b_k|, \qquad 0 < r < \delta. \tag{6}$$

From (5) and (6), we see that

$$|p(z)| \leq |b_0| - \frac{1}{2} r^k |b_k| < |b_0| \qquad \text{for } z = z_0 + re^{it}, 0 < r < \delta.$$

This contradiction proves the result. □

Second Version

Let P be a polynomial of positive degree with coefficients in $\mathbb{C}$. By Lemma 2 there exists $z_0 \in \mathbb{C}$ such that $|P(z_0)| \leq |P(z)|$ for all $z \in \mathbb{C}$. By considering the polynomial $p(z) = P(z + z_0)$, we may assume that $z_0 = 0$. We shall show that $P(0) = 0$.

There exists an integer $m \geq 1$ and $a, b \in \mathbb{C}$ with $b \neq 0$ such that

$$P(z) = a + bz^m + z^{m+1} Q(z), \tag{7}$$

where Q is a polynomial. Suppose $P(0) = a \neq 0$. Choose an mth root w of $-a/b$. Since w is fixed, the set $\{tw : 0 \leq t \leq 1\}$ is compact and since Q is continuous, there exists a constant M such that $\left|w^{m+1} Q(tw)\right| \leq M$ for all $0 \leq t \leq 1$. Hence we can choose $t \in (0, 1)$ such that $Mt < |a|$. Consequently for such a t we have

$$t \left|w^{m+1} Q(tw)\right| < |a|. \tag{8}$$

Now, we have, using (7),

$$\begin{aligned} P(tw) &= a + b(tw)^m + (tw)^{m+1} Q(tw) \\ &= (1 - t^m) a + (tw)^{m+1} Q(tw), \end{aligned} \tag{9}$$

so that

$$\begin{aligned} |P(tw)| &\leq (1 - t^m)|a| + tt^m \left|w^{m+1} Q(tw)\right| \\ &< (1 - t^m)|a| + t^m |a| \quad \text{by (8)} \\ &= |a| = P(0). \end{aligned}$$

This contradiction proves that $P(0) = 0$. □

Remark 4. A diligent reader would have observed that the above proof uses the following facts from **Analysis**:

(R1) Weierstrass Theorem: A real valued continuous function on a closed and bounded (i.e. compact) set attains maximum and minimum.

(R2) The simple equation $z^n = \alpha$ has a solution in $\mathbb{C}$ for $\alpha \in \mathbb{C}$. The standard proof of this fact uses the polar form of a complex number: If $\alpha = |\alpha|\, e^{i\theta}$, we take $z = |\alpha|^{1/n}\, e^{i\theta/n}$. This in turn depends on some properties of the transcendental functions such as the exponential function and trigonometric functions as well as the existence of nonnegative nth roots of nonnegative real numbers. Note that it requires quite a bit of analysis to construct the transcendental functions and establish their properties. However, see Remark 5 below which indicates a proof which avoids these results.

In fact in any proof of the fundamental theorem of algebra, some results from analysis or topology are needed. For instance, the standard so-called algebraic proofs use both of the following results from real analysis:

(R3) Every nonnegative real number has a nonnegative square root.

(R4) Every nonconstant polynomial of odd degree with real coefficients has a real root.

Hence the fundamental theorem of algebra is essentially a theorem in analysis which is needed in almost all branches of mathematics, including algebra and hence may be called a fundamental theorem of mathematics.

Remark 5. Following a note of Littlewood, we can avoid the use of the existence the m-th root of a complex number as follows. Let us observe that we may assume that m is odd, as the square roots always exist in $\mathbb{C}$. First of all notice that if $n = 2$, the result is well-known. For if $z = (x + iy)$ is a possible solution of $z^2 = a + ib$, we then have $x^2 - y^2 = a$ and $2xy = b$. We eliminate y from these equations to get $x^2 - (b^2/4x^2) - a = 0$ or

$$(x^2 - a/2)^2 = (b^2 + a^2)/4.$$

We may take $x^2 - a/2$ to be the unique nonnegative square root of $(b^2 + a^2)/4$. Finally, we take x to be the nonnegative square root of the nonnegative real number $(\sqrt{(a^2 + b^2)} - a)/2$. (Observe that we use (R3).) In view of this, we need only solve the equation $z^n = \lambda$ when n is odd. So we assume that n is odd in what follows.

Consider $f(z) = z^m - (a + ib)$. By Lemma 2, there exists z_0 such that $|f|$ attains its minimum at z_0. Write $f(z_0 + h) = A_0 + A_1 h + \cdots + h^m$. If $A_0 = f(z_0) = 0$, we are through. If $A_0 \neq 0$, and $z_0 \neq 0$ then $A_1 = m z_0^{m-1} h \neq 0$. We take $h = -t(A_0/A_1)$ and argue as in the last part of the proof above to show that there exists $0 < t < 1$ such that $|f(z_0 + h)| < |f(z_0)|$.

If $z_0 = 0$, and $a \neq 0$, then for $t \geq 0$, $\operatorname{Im} f(\pm t) = -b = \operatorname{Im} f(0)$ and $R := \operatorname{Re} f(\pm t) = (-1)^m t^m - a$. Since m is odd, one of the two signs must make $|R| < |a|$ for sufficiently small t. Hence $|f(z)| < |a + ib|$ where z is one of $\pm t$. If $a = 0$ and $b \neq 0$, we argue similarly with $f(\pm it)$.

(Geometric Proof: If you draw picture of $f(z_0) \neq 0$ and 0 and locate the points $f(\pm i\delta)$ and $f(\pm\delta)$, you can see one of these four points must be nearer to 0 than $f(z_0)$.)

3 Elementary Proofs using Leibniz Rule or Fubini

In this section we give two proofs, the first using Leibniz rule of differentiaton under the integral sign and the other using Fubini's theorem. The first one is due to Peter Loya. Let us recall the Leibniz's rule of differentiation under the integral sign: If $f(x,t)$ and $\frac{\partial f}{\partial x}(x,t)$ are continuous functions on $[a,b] \times [c,d]$, then $F(x) := \int_c^d f(x,t)\, dt$ is differentiable on $[a,b]$ and we have

$$F'(x) = \int_c^d \frac{\partial f}{\partial x}(x,t)\, dt.$$

Let $P(z) := z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be a polynomial with $a_j \in \mathbb{C}$, $0 \le j \le n-1$. Assume that P does not vanish at any point $\mathbb{C}$. For $r,t \in \mathbb{R}$, we define

$$f(r,t) := \frac{1}{P(re^{it})} = \frac{1}{r^n e^{int} + \cdots + a_1 re^{it} + a_0}.$$

By our assumption on P, $f(r,t)$ is an infinitely differentiable function of r and t. We define

$$F(r) := \int_0^{2\pi} f(r,t)\, dt.$$

We plan to show that F is a constant. This will lead to a contradiction, since $F(r) = F(0) = a_0^{-1} \cdot 2\pi$ is a nonzero constant. (Note that $a_0 = P(0)$ cannot be zero!) On the other hand, $F(r) \to 0$ as $r \to \infty$ thanks to the fact that $|P(z)| \to \infty$ as $|z| \to \infty$.

Now, we show that $F'(r) = 0$. By Leibniz rule, we have

$$F'(r) := \partial_r F(r) = \int_0^{2\pi} \partial_r f(r,t)\, dt. \tag{10}$$

We find

$$\frac{\partial f}{\partial r}(r,t) = -\frac{nr^{n-1}e^{int} + \cdots + a_1 e^{it}}{(r^n e^{int} + \cdots + a_1 re^{it} + a_0)^2}. \tag{11}$$

On the other hand, if we differentiate $f(r,t)$ with respect to t, we find that

$$\frac{\partial f}{\partial t}(r,t) = -\frac{i \cdot nr^n e^{int} + \cdots + ir \cdot a_1 e^{it}}{(r^n e^{int} + \cdots + a_1 re^{it} + a_0)^2} = ir \frac{\partial f}{\partial r}(r,t). \tag{12}$$

From (10)–(12), we conclude that

$$F'(r) = \frac{1}{ir}\int_0^{2\pi} \frac{\partial f}{\partial t}(r,t)\, dt = \frac{1}{ir} f(r,t)\Big|_0^{2\pi} = \frac{1}{ir}\left[\frac{1}{P(re^{i2\pi}) - P(re^{i0})}\right] = 0. \tag{13}$$

This completes the proof. □

We now give a proof which uses Fubini theorem.

Fubini theorem.

Let P be a complex polynomial which does not vanish on $\mathbb{C}$ and set $f := 1/P$. By continuity of f at 0, we have

$$\lim_{r \searrow 0} f(re^{it}) = f(0) = 0(\text{uniformly in } t \text{ on } \mathbb{R}). \tag{14}$$

By the chain rule, we get

$$D_\rho f(\rho e^{i\theta}) = e^{i\theta} f'(\rho e^{i\theta}) \text{ and } D_\theta f(\rho e^{i\theta}) = \rho i e^{i\theta} f'(\rho e^{i\theta}),$$

where $f'(z)$ denotes the complex derivative of f. As a consequence, we have the Cauchy Riemann equation in polar form:

$$D_\rho f(\rho e^{i\theta}) = \frac{1}{\rho i} D_\theta f(\rho e^{i\theta}). \tag{15}$$

For $0 < r < R < \infty$, we have the iterated integrals

$$\int_{-\pi}^{\pi} \int_r^R D_\theta f(\rho e^{i\theta}) d\rho \, d\theta \quad = \quad \int_{-\pi}^{\pi} [f(Re^{i\theta}) - f(re^{i\theta}) \, d\theta \tag{16}$$

$$\int_r^R \int_{-\pi}^{\pi} \frac{1}{\rho i} D_\theta f(\rho e^{i\theta}) \, d\theta d\rho \quad = \quad \int_r^R \frac{1}{\rho i} [f(\rho e^{i\pi}) - f(\rho e^{-i\pi}) \, d\rho = 0. \tag{17}$$

Since the function in (15) is continuous on the rcetangle $[-R, R] \times [-\pi, \pi]$, Fubini can be applied. Hence it follows from (16)–(17) that

$$\int_{-\pi}^{\pi} [f(Re^{i\theta}) - f(re^{i\theta}) \, d\theta = 0, \qquad \text{for } 0 < r < R < \infty. \tag{18}$$

If f were not a constant, $f(Re^{i\theta}) \to 0$ uniformly in θ as $R \to \infty$. By taking $r = 1/R$ and letting $R \nearrow \infty$, we deduce from (14) and (18)

$$\int_{-\pi}^{\pi} [0 - f(0)] \, d\theta = 0.$$

Consequently, $f(0) = 0$, a contradiction. □

Remark 6. The reader is recommended to compare the last two proofs keeping in mind that the version of Leibniz rule used above is deduced from Fubini. More specifically, note that (12) is the Cauchy-Riemann equations in polar form. Conclude that both the proofs are essentially the same!

4 Proofs that use Cauchy's Theorem from Complex Analysis

While there are quite a few proofs of FTA using complex analysis, the two proofs below use nothing more than the Cauchy's theorem.

Let us assume that P is a polynomial of degree is at least 2 and with real coefficients. Let Γ be the path $L_R \cup C_R$ where L_R is the line segment from $-R$ to R and C_R is the semi-circle from R to $-R$ in the upper half plane.

If $P(z)$ is never zero on $\mathbb{C}$, then $f(z) := 1/P(z)$ is entire on $\mathbb{C}$. Hence by Cauchy's theorem, $\int_{\Gamma_R} f(z) = 0$. When $R \to \infty$, $\int_{C_R} f(z) \to 0$, by the using the fact that $|P(z)| \to \infty$ as $|z| \to \infty$. It therefore follows that $\int_{-R}^{R} f(z) = 0$ for all $R > 0$. In other words, $\int_{\mathbb{R}} f(z) = 0$. Since $P(z) \in \mathbb{R}$ if $z \in \mathbb{R}$, and if we deduce from the vanishing of the integral $\int_{\mathbb{R}} f(z) = 0$ that f must change sign in $\mathbb{R}$ and has a real root, a contradiction.

To treat the general case, we apply the foregoing argument to the polynomial $Q(z) := P(z)\cdot\overline{P}(z)$ where $\overline{P}(z) := \sum_{k=0}^{n} \overline{a_k} z^k$ if $P(z) := \sum_{k=0}^{n} a_k z^k$. Clearly, $Q(z)$ had real coefficients.

Note that $Q(z) := (\sum_{r=0}^{n} a_r z^r)(\sum_{s=0}^{n} a_s z^s)$ so that

$$Q(z) = \sum_{k=0}^{2n} \sum_{r=0}^{k} (a_r \overline{a}_{k-r} + \overline{a}_r a_{k-r}) z^k = \sum_{k=0}^{2n} b_k z^k,$$

where $b_k := \sum_{r=0}^{k} (a_r \overline{a}_{k-r} + \overline{a}_r a_{k-r}) \in \mathbb{R}$.

We conclude that Q and hence P has a zero in $\mathbb{C}$. $\square$

Yet another proof using Cauchy's theorem runs as follows. Write $P(z) = zQ(z) + a_0$, $n \geq 1$ and $a_n \neq 0$. Then

$$\frac{1}{z} = \frac{P(z)}{zP(z)} = \frac{zQ(z) + a_0}{zP(z)} = \frac{Q(z)}{P(z)} + \frac{a_0}{zP(z)}.$$

If $P(z) \neq 0$ for all $z \in \mathbb{C}$, then $1/P$ is entire in $\mathbb{C}$. Let $\gamma_R(t) := Re^{it}$ for $0 \leq t \leq 2\pi$ and $R > 0$. By Cauchy's theorem $\int_{\gamma_R} \frac{Q}{P} = 0$ so that

$$2\pi i = \int_{\gamma_R} \frac{1}{z} = \int_{\gamma_R} \frac{a_0}{zP(z)}.$$

By Lemma 1, we have $|P(z)| \geq \frac{1}{2}|a_n||z|^n$ for large R. Hence it follows that

$$2\pi = |\int_{\gamma_R} \frac{a_0}{zP(z)}\, dz| \leq 2\pi R \frac{2|a_0|}{R|a_n|R^n} = \frac{4\pi|a_0|}{|a_n|R^n} \to 0,$$

as $R \to \infty$. Thus, we get the contradiction: $2\pi = 0$. We therefore conclude that our assumption that $P(z) \neq 0$ for all $z \in \mathbb{C}$ is wrong. $\square$

A third proof of FTA, due to R.P. Boas is given below. The proof is very interesting for it dose not use the growth behaviour of a polynomial at infinity!

Assume that the polynomial P takes real values on $\mathbb{R}$. Since $P(z)$ never vanishes, $P(2\cos t) \neq 0$ for any $t \in \mathbb{R}$. Consider the integral

$$I := \int_0^{2\pi} \frac{dt}{P(2\cos t)}.$$

Since P has no zeros and $P(x) \in \mathbb{R}$ for $x \in \mathbb{R}$, it follows that $P(2\cos t)$ has the same sign for $t \in [0, 2\pi]$. Hence, we conclude that $I \neq 0$.

We now interpret the integral I as an integral along the unit circle $\gamma(t) := e^{it}$ for $0 \le t \le 2\pi$:

$$I = -i \int_\gamma \frac{dz}{zP(z+z^{-1})}.$$

If $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ with $a_n \neq 0$, then

$$P(z+1/z) = a_0 + a_1 \left(z + \frac{1}{z}\right) + \cdots + a_n \left(z + \frac{1}{z}\right)^n = z^{-n} Q(z),$$

where $Q(z)$ is some polynomial. Note that $Q(0) = a_n \neq 0$ and $Q(z)$ has no zeros in $\mathbb{C}$. Hence the integrand in I is $z^{n-1}/Q(z)$ which is holomorphic in $\mathbb{C}$. By Cauchy's theorem, $I = 0$. This contradicts our earlier conclusion. $\square$

A fourth proof which uses the fact that an entire function can be represented by a power series is given below.

Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a complex polynomial of degree $n \ge 1$. If it has no zeros in $\mathbb{C}$, then $f(z) = 1/P(z)$ is an entire function and hence can be represented by a power series on whole of $\mathbb{C}$: $f(z) = \sum_{k=0}^\infty b_k z^k$. The following observation is the key to the proof.

Lemma 7. *There exist positive real numbers c and r such that $|b_k| > cr^k$ for infinitely many k.* $\square$

Assume the lemma. If we take $z = 1/r$ in the power series for f, then for infinitely many k, we have

$$|b_k z^k| = |b_k| r^{-k} > c > 0.$$

Thus the k-th terms of the convergent series for $f(1/r)$ does not approach 0, a contradiction.

Proof of the lemma. We start with the fact that $1 = P(z)f(z)$ and equate the coefficients:

$$1 = P(z)f(z) = (a_0 + a_1 z + \cdots + a_n z^n)(b_0 + b_1 + \cdots + c_k z^k + \cdots).$$

We have $a_0 b_0 = 1$ and hence $a_0 \neq 0$ and $b_0 \neq 0$. Also, by equating the coefficients of z^{n+k}, we find that

$$a_0 b_{k+n} + a_1 b_{k+n-1} + \cdots + a_n b_k = 0 \text{ for all } k \ge 0. \tag{19}$$

Since $b_0 \neq 0$, we choose c such that $0 < c < |b_0|$. Since $a_n \neq 0$, we choose an $r > 0$ such that

$$|a_0| r^n + |a_1| r^{n-1} + \cdots + |a_{n-1}| r \le |a_n|.$$

By our choice, $|b_0| > c = cr^0$. Now suppose that $|b_k| > cr^k$. We claim that for some j between 1 and n, we shall have $|b_{k+j}| > cr^{k+j}$. This claim proves the lemma. We shall prove the claim by contradiction. Thus, $|b_{k+j}| \leq cr^{k+j}$ for all $1 \leq j \leq n$. From (19), we obtain

$$\begin{aligned}
|b_k| &= \frac{|a_0 b_{k+n} + a_1 b_{k+n-1} + \cdots + a_{n-1} b_{k+1}|}{|a_n|} \\
&\leq \frac{|a_0||b_{k+n}| + |a_1||b_{k+n-1}| + \cdots + |a_{n-1}||b_{k+1}|}{|a_n|} \\
&\leq \frac{|a_0|cr^{k+n} + |a_1|cr^{k+n-1} + \cdots + |a_{n-1}|cr^{k+1}}{|a_n|} \\
&\leq cr^k \left(\frac{|a_0|r^n + |a_1|r^{n-1} + \cdots + |a_{n-1}|r}{|a_n|} \right) \\
&\leq cr^k.
\end{aligned}$$

This contradicts our assumption that $|b_k| > cr^k$. Hence the lemma is proved.

A fifth proof due to R.P. Boas uses Picard's theorem. If a nonconstant polynomial P does not take the value 0, we claim that it does not take at least one of the values $1/k$ for $k \in \mathbb{N}$. For, if it does, then there exist $z_k \in \mathbb{C}$ such that $P(z_k) = 1/k$ for all $k \in \mathbb{N}$. We claim that $\{z_k : k \in \mathbb{N}\}$ is a bounded set. For, since $|P(z)| \to \infty$ as $|z| \to \infty$, there exists $R > 0$ such that $|P(z)| > 1$ for all z with $|z| > R$. Hence we conclude that $|z_k| \leq R$ for all $k \in \mathbb{N}$. By Bolzano-Weierstrass theorem, there exist a convergent subsequence, say, (z_{k_n}) converging to, say, z_0. By continuity, it follows that

$$P(z_0) = \lim_n f(z_{k_n}) = \lim_n 1/k_n = 0,$$

a contradiction. Hence our claim is established. Thus our entire function does not take the value 0 and $1/k$ for some k. Invoking Picard's theorem, we conclude that P is a constant, a contradiction.

5 Consequences of FTA

The following are some of the standard consequences (mainly algebraic in nature) of the fundamental theorem of algebra.

Theorem 8. *Any nonconstant polynomial P of degree n in $\mathbb{C}$ has n zeros in $\mathbb{C}$. In fact, there exist complex numbers α_i, $1 \leq i \leq n$ and $\lambda \in \mathbb{C}$ such that $P(\alpha_i) = 0$ for $1 \leq i \leq n$ and $P(z) = \lambda(z - \alpha_1) \cdots (z - \alpha_n)$.*

Proof. For any $\alpha \in \mathbb{C}$, we can write:

$$P(z) = P((z-\alpha)+\alpha) = \sum_{k=0}^{n} a_k [(z-\alpha)+\alpha]^k = \sum_{k=0}^{n} \sum_{j=0}^{k} a_k \binom{k}{j} (z-\alpha)^j \alpha^{k-j}, \tag{20}$$

using binomial theorem. Each term on the right side of this equation for $j > 0$ has factor of $(z - \alpha)$. Hence we can write

$$P(z) = (z-\alpha) q_\alpha(z) + \sum_{k=0}^{n} a_k \alpha^k, \tag{21}$$

for some polynomial q_α of degree $n-1$. (q_α depends on α.)

By Lemma 2 and Theorem. 3, there is an $\alpha_1 \in \mathbb{C}$ such that $P(\alpha_1) = 0$. Replacing α by α_1 in (21), we get $P(z) = (z - \alpha_1)q_{\alpha_1}(z)$. Since q_{α_1} is a polynomial of degree $n-1$, the result follows by induction. □

Ex. 9. Show that the complex zeros of a *real* polynomial occur in conjugate pairs, that is, if $\alpha \in \mathbb{C}$ is a zero of $p(z) := \sum_{k=0}^{n} a_k z^k$ with $a_k \in \mathbb{R}$ for all $1 \le k \le n$, then $\overline{\alpha}$ is also a zero of p.

Ex. 10. Show that a real polynomial can always be expressed as a product of real polynomials of degree less than or equal to 2. Express $z^8 - 1$ in this way.

Ex. 11. Prove that for $z \in \mathbb{C}$, $z \neq 1$ we have $1 + z + \cdots + z^n = \frac{1-z^{n+1}}{1-z}$, for any $n \in \mathbb{Z}_+$. Hence deduce that

$$\frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} + \alpha^{n-2}\beta + \cdots + \alpha\beta^{n-2} + \beta^{n-1}, \qquad \text{for } \alpha \neq \beta.$$

Hint: Take $z = \alpha/\beta$ in the first part.

Ex. 12. Let $\alpha \in \mathbb{C}$ be given. A polynomial p is divisible by $(z-\alpha)$ if there exists a polynomial q such that $p(z) = (z-\alpha)q(z)$. For $n \in \mathbb{N}$ and $c \in \mathbb{C}$, show that $(z-\alpha)$ divides $c(z^n - \alpha^n)$. *Hint:* Exer. 11.

Ex. 13. If p is a polynomial and $z_0 \in \mathbb{C}$ is arbitrary, show that $p(z) - p(z_0)$ is divisible by $z - z_0$. *Hint:* Express $p(z) - p(z_0)$ as a sum of terms of the type $z^k - z_0^k$.

Ex. 14. Show that $p(z_0) = 0$ iff $p(z)$ is divisible by $(z - z_0)$.

Ex. 15. Let p be a polynomial of degree at most n. Assume that there exists n distinct points z_k, $1 \le k \le n$ such that $p(z_k) = 0$ for all k. Prove that $p(z) = c(z - z_1)\cdots(z - z_n)$ for some $c \in \mathbb{C}$. *Hint:* By Exer. 14, $p(z) = (z - z_n)q(z)$ and $q(z_k) = 0$ for $1 \le k \le n-1$. Use induction.

Ex. 16. If a polynomial of degree at most n vanishes at $n+1$ distinct points, then it is the zero polynomial. *Hint:* Put $z = z_{n+1}$ in the Exer. 15.

Ex. 17. If p and q are polynomials of degree at most n and if they agree at $n+1$ distinct points then $p = q$.