Fubini Theorem on $\mathbb{R}^n$

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Theorem 1 (Fubini)**.** *Let $f \in \mathcal{L}(\mathbb{R}^{k+l})$. Then we have*

$$\int_{R^{k+l}} f(x,y)\, d(x,y) = \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^l} f(x,y)\, dy \right) dx. \tag{1}$$

That is, if we set $F(x) := \int_{\mathbb{R}^l} f(x,y)\, dy$, then
i) $F(x)$ exists for almost all $x \in \mathbb{R}^k$,
ii) $\int_{\mathbb{R}^k} F(x)\, dx$ exists and
iii) $\int_{\mathbb{R}^k} F(x)\, dx = \int_{\mathbb{R}^{k+l}} f(x,y)\, d(x,y)$. In fact, we have

$$\int_{R^{k+l}} f(x,y)\, d(x,y) = \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^l} f(x,y)\, dy \right) dx = \int_{\mathbb{R}^l} \left(\int_{\mathbb{R}^k} f(x,y)\, dx \right) dy. \tag{2}$$

Note that "a.e." is to be understood in the respective spaces.

Proof. We prove the result in stages. We verify it for characteristic functions of bounded intervals, for step functions, for functions in $\mathcal{L}^+$ and then for elements of $\mathcal{L}$.

Step 1: The theorem is true for characteristic functions of (bounded) intervals. Let $K := I \times J$ be an interval in $\mathbb{R}^{k+l}$. Let $f = \chi_K$. Then, in an obvious notation,

$$\int_{\mathbb{R}^{k+l}} f(x,y) d(x,y) = \ell(K) = \prod_{i=1}^{k} (b_i - a_i) \cdot \prod_{j=1}^{l} (\beta_j - \alpha_j).$$

Now $F(x) = \int_{\mathbb{R}^l} f(x,y)\, dy = \int_{\mathbb{R}^l} \chi_I(x)\chi_J(y)\, dy = \ell(J)\chi_I(x)$ for *all* $x \in \mathbb{R}^k$. We have $\int_{\mathbb{R}^k} F(x)\, dx = \ell(J)\ell(I)$ and the result follows in this case.

Step 2: The result is true for any step function on $\mathbb{R}^{k+l}$. This follows from Step 1 and the fact that the integral on LHS and both the integrals on the RHS of Eq. 1 are linear. We note that in this case also $F(x)$ is defined for all $x \in \mathbb{R}^k$ and that it is a step function on $\mathbb{R}^k$.

Step 3: is to prove the following special case of Fubini theorem:

Proposition 2. *Let E be a null set (i.e., a set of measure zero) in $\mathbb{R}^{k+l}$. Fix $x \in \mathbb{R}^k$. Let $E_x := \{y \in \mathbb{R}^l : (x,y) \in E\}$. Then E_x is of measure zero for almost all $x \in \mathbb{R}^k$.*

Proof. First of all note that we cannot claim that E_x is of measure zero for *all* $x \in \mathbb{R}^k$. For example, take $E := \{(x, y) \in \mathbb{R}^2 : x \in \mathbf{Q}, y \in \mathbb{R}\}$.

To prove this special case of the theorem, let $\{g_n\}$ be an increasing sequence of step functions on $\mathbb{R}^{k+l}$ such that $\{\int g_n\}$ converges but $\{g_n(x, y)\}$ diverges at all points of E. Let $G_n(x) := \int g_n(x, y)\,dy$. Then $G_n(x)$ is defined for all $x \in \mathbb{R}^k$, it is a step function and it satisfies $\int G_n(x)\,dx = \int g_n(x, y)\,d(x, y)$ by Step 2. Hence by one of fundamental lemmas $\lim_n G_n(x)$ exists for almost all x, say, for all $x \in G$. Note that $\mathbb{R}^k \setminus G$ is a null set. Let x_0 be one such point, i.e., $\lim_n \int g_n(x_0, y)\,dy$ exists. Then again by the fundamental lemma, we see that $\lim_n g_n(x_0, y)$ exists for almost all $y \in \mathbb{R}^l$. But, by our choice, $\lim_n g_n(x_0, y)$ diverges for all $y \in E_{x_0}$. The last two statements imply that E_{x_0} is of measure zero. As x_0 is an arbitrary point of G, E_x is of measure zero for all points in G. □

Step 4: Let $f \in \mathcal{L}^+(\mathbb{R}^{k+l})$. Let $f_n \in \mathcal{S}(\mathbb{R}^{k+l})$ be such that f_n increases with $\lim_n f_n(x, y) = f(x, y)$ for almost all $(x, y) \in \mathbb{R}^{k+l}$ and $\int f := \lim_n \int f_n(x, y) d(x, y)$. Then F_n, defined as above, is a step function defined for all $x \in \mathbb{R}^k$. From Step 2, we have $\int f_n(x, y)\,d(x, y) = \int F_n(x)\,dx$. Since $f_n \leq f_{n+1}$, we have $F_n \leq F_{n+1}$ by positivity of the integral on the step functions on $\mathbb{R}^l$. Now the fact that $\{\int f_n(x, y)\,d(x, y)\}$ is convergent implies that $\{\int F_n\}$ is convergent. Hence the fundamental lemma implies that $\lim_n F_n(x)$ exists for almost all $x \in \mathbb{R}^k$.

Let $E := \{(x, y) \in \mathbb{R}^{k+l} : \lim_n f_n(x, y) \neq f(x, y)\}$. By our choice of f_n, E is of measure zero. By the Proposition in Step 3, E_x is of measure zero for almost all $x \in \mathbb{R}^k$. Thus for almost all $x \in \mathbb{R}^k$ both the following statements are true: i) $\lim_n F_n(x)$ exists and ii) E_x is of measure zero. Let $x_0 \in \mathbb{R}^k$ be such that both the statements are true. Hence $\lim_n f_n(x_0, y) = f(x_0, y)$ for almost all $y \in \mathbb{R}^l$ and $\lim_n F_n(x_0) = \lim \int f_n(x_0, y)\,dy$ exists. It follows that $\int f(x_0, , y)\,dy$ exists and equals $\lim F_n(x_0)$.

We set $F(x_0) := \int f(x_0, y)\,dy = \lim_n F_n(x_0)$ and $F(x) = 0$ otherwise. The increasing sequence $\{F_n\}$ converges to F almost everywhere on $\mathbb{R}^k$ and

$$\lim \int F_n(x)\,dx = \lim_n \int f_n(x, y)\,d(x, y) = \int f(x, y)\,d(x, y).$$

Thus $F \in \mathcal{L}^+(\mathbb{R}^k)$ with $\int_{\mathbb{R}^k} F(x)\,dx = \int_{\mathbb{R}^{k+l}} f(x, y)\,d(x, y)$.

Step 5: The result extends by linearity to $f \in \mathcal{L}(\mathbb{R}^{k+l})$.

Step 6: Interchanging the roles of x and y, we get

$$\int_{R^{k+l}} f(x, y)\,d(x, y) = \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^l} f(x, y)\,dy \right) dx = \int_{\mathbb{R}^l} \left(\int_{\mathbb{R}^k} f(x, y)\,dx \right) dy.$$

This completes the proof of the Fubini theorem. □

For any functions φ and $\psi \geq 0$, the *truncation* of φ by ψ, in notation $\mathrm{mid}(-\psi, \varphi, \psi)$, is defined by

$$\mathrm{mid}(-\psi, \varphi, \psi)(x) := \begin{cases} \min\{\psi(x), \varphi(x)\} & \text{if } \varphi(x) \geq 0 \\ \max\{-\psi(x), \varphi(x)\} & \text{if } \varphi(x) \leq 0. \end{cases}$$

We need the following two corollaries of DCT.

Lemma 3. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be such that there exists a sequence $f_n \in \mathcal{L}$ such that* $\lim f_n = f$ *a.e. Assume further that $|f| \leq g$ for some $g \in \mathcal{L}$. Then $f \in \mathcal{L}$.*

Proof. Note the difference in hypothesis of DCT and this lemma. In DCT, the f_n's are dominated whereas here f is dominated.

Let h_n be the truncation of f_n by g, i.e., $h_n := \max\{-g, \min\{f_n, g\}\}$. Then $h_n \in \mathcal{L}$. Clearly, $\lim h_n = f$ a.e. and $|h_n| \leq g$. Hence DCT can be applied to get the result. □

The above also suggests a new concept:

f is said to be *measurable* iff for any bounded interval I and any nonnegative constant c the *truncated* function $\mathrm{mid}(-c\chi_I, f, c\chi_I)$ is in $\mathcal{L}$. An equivalent condition for measurability of f is that the truncation of f by any nonnegative integrable function φ, viz., $\mathrm{mid}(-\varphi, f, \varphi)$ is integrable. (Exercise: Prove the equivalence of these two definitions.)

An important fact we need below, which follows from the corollary of the DCT, is

Proposition 4. *Let f be measurable and assume that there exists an integrable function g such that $|f| \leq g$ a.e., then f is integrable. In particular, if f is measurable and $|f|$ is integrable, then f is integrable.*

Proof. Just by definition: For, measurability means that $\mathrm{mid}(-g, f, g) \equiv f$ is integrable! □

A partial converse to Fubini's theorem is Tonelli's theorem. It is perhaps more useful in practice than Fubini's theorem.

Theorem 5 (Tonelli)**.** *Let $f : \mathbb{R}^{k+l} \to \mathbb{R}$ be a measurable function. Assume that one of the repeated integrals of $|f|$ exists. Then f is integrable and we have*

$$\int_{R^{k+l}} f(x,y)\, d(x,y) = \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^l} f(x,y)\, dy \right) dx = \int_{\mathbb{R}^l} \left(\int_{\mathbb{R}^k} f(x,y)\, dx \right) dy.ubini3$$

Proof. We assume that $\int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^l} |f|(x,y)\, dy \right) dx$ exists.

Since f is measurable so is $|f|$. We let I_n to be the product of intervals of the form $[-n, n]$ in $\mathbb{R}^k, \mathbb{R}^l, \mathbb{R}^{k+l}$. We also let $\psi_n(x,y)$ be n-times the characteristic function of I_n in the product space. Then, by measurability of $|f|$, we have $f_n := \mathrm{mid}(-\psi_n, |f|, \psi_n) \in \mathcal{L}(\mathbb{R}^{k+l})$. Clearly f_n increases and converges to $|f|$. Fubini theorem applied to f_n yields that $\int_{\mathbb{R}^{k+l}} f_n(x,y)\, d(x,y) = \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^l} f_n(x,y)\, dy \right) dx = \int_{\mathbb{R}^l} \left(\int_{\mathbb{R}^k} f_n(x,y)\, dx \right) dy$. Thus the MCT implies that $|f|$ is integrable. Now Proposition 4 finishes the proof. □