Illustration of Gauss Theorem in Theory of Surfaces

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Theorem 1 (Gauss Golden Theorem)**.** *If S_1 and S_2 are two surfaces in $\mathbb{R}^3$ and if $\varphi\colon S_1 \to S_2$ is a local isometry, then the Gaussian curvature K_1 and K_2 are the same at corresponding points. In particular, K does not depend on the way it is embedded in $\mathbb{R}^3$ but only on the first fundamental form, i.e., on the induced inner products on the tangent spaces.*

We explain the meaning of this by means of examples.

Example 2. Consider

$$S_1 = \{p \in \mathbb{R}^3 : z(p) = 0\}$$

and

$$S_2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\},$$

the cylinder. Let $\varphi\colon S_1 \to S_2$ be given by

$$(x, y, 0) \mapsto (\cos x, \sin x, y).$$

Then we have seen that φ is a local isometry. Clearly $K_1 = 0 = K_2$ at corresponding points. If we let

$$S = \{(x, y, 0) \in \mathbb{R}^3 : -\pi < x < \pi, y \in \mathbb{R}\} \subset \mathbb{R}^3$$

and $S_1^1 = S$ and $S_2^1 =$ the image of S_1^1 under φ i.e., the cylinder with a line removed (parallel to the z-axis), then S_i^1 is an embedding of S into $\mathbb{R}^3$ in two different ways. But however the first fundamental forms at the corresponding points are the same and hence by Gauss theorem we should expect $K_1 = K_2$ etc.

Example 3. As a second example consider the local isometry φ between the punctured plane $S = \mathbb{R}^2 \setminus \{0\}$ and the cone $\widetilde{S} = \{(x, y, z) \in \mathbb{R}^3 : 3x^2 + 3y^2 = z^2, z > 0\}$ given by

$$\varphi(r\cos\theta, r\sin\theta) = (\frac{r}{2}\cos 2\theta, \frac{r}{2}\sin 2\theta, \frac{\sqrt{3}}{2}r).$$

We leave it to the reader to check that the Gaussian curvatures coincide at the corresponding points.

Example 4. Helicoid and Catenoid Consider the Helicoid (spiral staircase) given by

$$\varphi(u, v) = (u\cos v, u\sin v, v)$$

with $u > 0$ and $v \in \mathbb{R}$ and the Catenoid given by

$$\psi(\theta, \phi) \mapsto (\cosh\theta\cos\phi, \cosh\theta\sin\phi, \theta)$$

with $0 < \phi < 2\pi$ and $\theta \in \mathbb{R}$, and the local isometry given by $v \mapsto \phi$, $u \mapsto \sinh\theta$. The reader should calculate I, II, K etc., and verify the theorem in this case.