Geodesics on a Sphere

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

We give two proofs of the well-known fact that the only (unit speed) geodesics on a sphere are great circles with standard parametrization.

Let S denote the unit sphere $S(0,1)$ in $\mathbb{R}^3$ and c be a unit speed geodesic on S. If we write $c(s) = (x(s), y(s), z(s))$, then $c''(s)$ must be a scalar multiple of the normal and hence is of the form $\lambda(s)c(s)$, for some $\lambda(s) \in \mathbb{R}$. We differentiate the identity $\langle c(s), c'(s)\rangle = 0$ and obtain

$$\begin{aligned} 0 &= \left\langle c'(s), c'(s) \right\rangle + \left\langle c(s), c''(s) \right\rangle \\ &= 1 + \langle c(s), \lambda(s)c(s)\rangle \\ &= 1 + \lambda(s). \end{aligned}$$

Thus we see that $\lambda(s) = -1$ for all s. then we get a contradiction. Hence we conclude that $\lambda(s) = -1$ for all s. We have therefore arrived at the following result. If $c(s) = (x(s), y(s), z(s))$ is a unit speed geodesic, then we get a system of ODE's:

$$x''(s) = -x(s), \;\; y''(s) = -y(s) \text{ and } z''(s) = -z(s).$$

If we assume that $c(0) = p = (x_0, y_0, z_0)$ and $c'(0) = (x'(0), y'(0), z'(0)) = (v_1, v_2, v_3)$, then we have arrived the following system of linear ODE with the initial conditions:

$$\begin{aligned} x''(s) &= -x(s), \quad x(0) = x_0, \quad x'(0) = v_1 \\ y''(s) &= -y(s), \quad y(0) = y_0, \quad y'(0) = v_2 \\ z''(s) &= -z(s), \quad z(0) = z_0, \quad z'(0) = v_3. \end{aligned}$$

The solutions are given by $x(s) = x_0 \cos s + v_1 \sin s$ and so on. In other words, $c(s) = (\cos s)p + (\sin s)v$. □

Ex. 1. Adapt the argument above to show that the unit speed geodesics on $S(0,R)$ are of the form

$$c(s) := \cos(s/R)p + R\sin(s/R)v,$$

where $p \in S(0,R)$ and v s unit vector perpendicular to p.

We now give a second proof of the result.

Lemma 2. *Let c be a unit speed curve in $\mathbb{R}^3$ with constant curvature κ and torsion $\tau = 0$. Then c is part of a circle with radius $1/\kappa$.*

Proof. Since $\tau = 0$, the curve c lies in a plane. Our strategy is to show that $c(s)$ is at a distance $1/\kappa$ from some fixed point p. Consider the curve $\gamma(s) := c(s) + \mathbf{n}/\kappa$. Using the hypothesis and the Frenet formulas, we get

$$\gamma' = c' + \frac{1}{\kappa}\mathbf{n}' = \mathbf{t} + \frac{1}{\kappa}(-\kappa\mathbf{t}) = 0.$$

Hence the curve γ is a constant, that is, $\gamma(s) = c(s) + \frac{1}{\kappa}\mathbf{n}(s) = p$. We observe that

$$d(c(s), p) = \|c(s) - p\| = \left\| \frac{1}{\kappa}\mathbf{n}(s) \right\| = 1/\kappa.$$

This proves the lemma. □

Observation. Let c be a unit speed geodesic on a surface $S \subset \mathbb{R}^3$. Assume that the curvature of c, as a curve in $\mathbb{R}^3$, is positive. Note that the principal normal $\mathbf{n} = c''/\kappa$ is normal to S. By the definition of the Weingarten map, we have

$$-\mathbf{n}' = L(c') = L(\mathbf{t}).$$

But from the Frenet equations, we have $\mathbf{n}' = \tau\mathbf{b} - \kappa\mathbf{t}$. Hence we conclude that

$$L(\mathbf{t}) = \kappa\mathbf{t} - \tau\mathbf{b}. \tag{1}$$

□

We now show that any unit speed geodesic on the sphere $S = S(0, r)$ is a (part of a) great circle. Let c be a unit speed geodesic on S. We have already seen that the Weingarten map L on S is given by $L_p = \frac{1}{r}I$. In particular,

$$L(\mathbf{t}) = \frac{\mathbf{t}}{r}. \tag{2}$$

From (1)–(2), we conclude that $\kappa = 1/r$ and $\tau = 0$. By Lemma 2, we deduce that c is part of a circle lying in a plane. Since c is part of a circle (which is the intersection of a plane and the sphere S) of radius r, we infer that c must lie on a plane through the origin. (See the Fact below.)

Fact. Let P be a plane given by $ax + by + cz = d$. If P intersects $S(0, r)$, then the intersection is a circle of radius $0 \le \rho \le r$. Furthermore, $\rho = r$ iff $d = 0$, that is, P is plane passing through the origin.

Since $(a, b, c) \neq 0$, we may assume that it is a unit vector. By composing with a rotation, we may assume that $(a, b, c) = e_3$. Then the plane is $z = d$. Now, the intersection of this plane with the sphere is the set of points (x, y, d) with $x^2 + y^2 + d^2 = r^2$. The fact follows from this. □