Group Actions

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Definition 1. Let X be a set, G a group. A *group action* of G on X is a map $\alpha : G \times X \to X$ given by
$\alpha(ab, x) = \alpha(a, \alpha(b, x))$ for all $a, b \in G$, $x \in X$.
$\alpha(e, x) = x$ for all $x \in X$.

We usually drop α and write $\alpha(g, x)$ as $g \cdot x$ or gx. Then (1) reads: $(ab) \cdot (x) = a \cdot (b \cdot x)$. We also say G *acts on* X and X is a G–set (when the action α is understood).

The *orbit* of G in X is a set of the form $G \cdot x = \{g \cdot x \mid g \in G\}$ for a fixed $x \in X$. $G \cdot x$ is also called the *orbit of* x, denoted by $\mathcal{O}_x$. Note that $y \in Gx$ iff $y = gx$ for some $g \in G$ iff $x = g^{-1}y$ for some $g \in G$ iff $x \in Gy$. Thus $Gx = Gy$ iff $\mathcal{O}_x = \mathcal{O}_y$. Define an equivalence relation $x \sim y$ iff $Gx = Gy$. Its equivalence classes are orbits of G in X and X is the disjoint union of orbits of G.

Example 2. $GL(n, \mathbb{R}) = \{n \times n \text{ invertible matrices }\}$ acts on $\mathbb{R}^n$.

Example 3. $O(n, \mathbb{R}) = \{n \times n \text{ orthogonal matrices }\}$ acts on $\mathbb{R}^n$.

Example 4. If X is any set, S_X, the symmetry group (of all bijections of X) acts on X.

Example 5. Any group G acts on itself via *left action:* $X = G$, $G \times X \to X$ given by $(g, x) \mapsto g \cdot x$, the group multiplication.

Example 6. A group G acts on itself via *conjugation:* $(g, x) \mapsto gxg^{-1}$. The orbits are called *conjugacy classes.*

Ex. 7. Find the orbits in the above examples.

Ex. 8. What are the orbits in Example 6 if G is abelian?

Example 9. Let $H \leq G$. Let H act on $X = G$ via the left action: $(h, x) \mapsto hx$. The orbits are of the form Hx for some $x \in G$. Note that $Hx = Hy$ iff $xy^{-1} \in H$. Write $G = \cup_{x \in G} Hx$. Any two orbits are bijective via the map $h \to hx$. If G is finite we deduce Lagrange's theorem.

Definition 10. G acts *transitively* on X if for any $x, y \in X$, there is a $g \in G$ such that $gx = y$. That is, there is only one orbit in X.

Ex. 11. Which of the actions in Example 2 to Example 9 are transitive?

Definition 12. Fix $x \in X$. Let $G_x := \{g \in G \mid gx = x\}$. Then G_x is called the *stabiliser* of x in G and is a subgroup. G_x is also called the *isotropy* subgroup of x.

Ex. 13. Let $\mathcal{O}_x = \mathcal{O}_y$. How are the stabilisers G_x and G_y related?

Ex. 14. Find the stabilisers of various elements in the above six examples.

Ex. 15. Consider $\mathbb{R}^n$ as the vector space of colum vector $n \times 1$. Let $GL_n(\mathbb{R})$ act on $\mathbb{R}^n$ as follows: $(A, x) \mapsto Ax$, the matrix multiplication. Show that the isotropy of e_n is the subgroup of the elments of the form $\begin{pmatrix} A & 0 \\ x & 1 \end{pmatrix}$ where $A \in GL_{n-1}$ and $x \in \mathbb{R}^{n-1}$.

Ex. 16. Let $V : \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x_n = y_n\}$. Let GL_n act on V as follows:

$$A(x, y) = (x^t A, Ay).$$

What is the isotropy at (e_n, e_n)?

Ex. 17. Every subgroup H of a group G occurs as a stabiliser of an element in a G–set. *Hint:* Let $X = \{Hg \mid g \in G\}$ be the set of (right) cosets, i.e., the orbits of H in G with respect to the left action of H on G. Then $|X| = [G : H]$, the index of H in G. G acts on X by $(a, Hg) \mapsto Hga^{-1}$. (If Y is the set of left cosets, i.e., $X = \{gH \mid g \in G\}$, then G on Y by $(a, gH) \mapsto agH$). This action is transitive. If $x = H \in X$, then the stabiliser of x is H.

Definition 18. Let X and Y be two G–sets. Then these are *G–isomorphic* if there is a bijection $f : X \to Y$ such that $g \cdot f(x) = f(g \cdot x)$ for all $g \in G$, $x \in X$. Draw a commutative diagram.

Ex. 19. Let G act transitively on X. Then X is G–isomorphic to G/H (the set of left cosets of H in G) for some subgroup $H \leq G$. *Hint:* H is the stabiliser of a fixed $x \in X$.

Ex. 20. $|Gx| = [G : G_x]$. (G acts on X, $x \in X$).

Definition 21. Let G act on itself via conjugation. Then $G_x = \{g \in G \mid gx = xg\}$ is known as the *centraliser* $Z_x(G)$ of $x \in G$. $G \cdot x = C_x$ is called the *conjugacy class* of x in G. Note that $|C_x|$ is a divisor of $|G|$ if G is finite. If G is finite and $\{C_1, \ldots, C_r\}$ are disjoint conjugacy classes, then $|G| = |C_1| + \cdots + |C_r|$ is called the *class equation.* Note that $G = |Z_G| + \sum_{i=1}^{r} [G : G_{x_i}]$ if $\{G \cdot x_i\}$ are the distinct conjugacy classes of G with $|G \cdot x_i| > 1$.

Ex. 22. Use the class equation to prove that if $|G| = p^r$, then $Z(G)$, the center of G is non-trivial.

Ex. 23. Let G act on X and Y. Define a G action on $X \times Y$ in an obvious way. Relate the stabiliser of (x, y) with those of x and y.

Ex. 24. Let G be a group and let H, K be subgroups of index r and s respectively. Show that $H \cap K$ has index at most rs.

Ex. 25. Let G be a group, H, K subgroups of G of index r. Assume H and K are conjugate. Show that $H \cap K$ has index at most $r(r-1)$.

Ex. 26. Let $\{C_i\}$ be the conjugacy classes in a group. Show that each product $C_i C_j$ is a union of conjugacy classes.

Ex. 27. Determine all groups with only two conjugacy classes.

Ex. 28. Let $H \leq G$. Let $X = \{xH \mid x \in G\}$. Let G act on X by $(g, xH) \mapsto gxH$. Prove that H is a normal subgroup of G iff every H–orbit in X is a singleton.

Ex. 29. Let $|G| < \infty$ and let p be the smallest prime dividing $|G|$. Let $H \leq G$ such that $[G : H] = p$. Show that H is a normal subgroup of G.

Remark 30. Recall the class equation: G acts on itself via conjugation. Therefore $|G| = |\mathcal{O}_1| + \cdots |\mathcal{O}_r|$, $|O_i| = \left\{gxg^{-1} \mid g \in G\right\} = \mathcal{O}_x$. $|G| := |Z(G)| + \sum_{i=1}^{r} |[G : C_G(x)]|$ in classical notation.

Ex. 31. Let p be a prime and $|G| = p^n$. Then G has a non-trivial centre. *Hint:* Apply class equation. Note that $x \in Z(G)$ iff $|\mathcal{O}_x| = 1$.

Ex. 32. If p is a prime and $|G| = p^2$, then $G \simeq \mathbb{Z}/p^2$ or $\mathbb{Z}/p \times \mathbb{Z}/p$.

Ex. 33. Let $H \leq G$ with $[G : H] = n$. Show that H contains a normal subgroup K of G such that $[G : K] \leq n!$.

Ex. 34. Let G be a simple group (i.e., having no proper normal subgroup) with a subgroup H of finite index $n > 1$. Show that $|G| \leq n!$.