Fundamental Theorem of Calculus in HK-Integral

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

Abstract

The aim of this article is to bring to the attention of college teachers a generalized Riemann integral called Henstock-Kurzweil integral. This has attracted a growing interest in recent years. New introductory textbooks on real analysis have a chapter on this topic. Leading American universities have started teaching this to undergraduate students, after a recommendation by a Committee appointed by the National Science Foundation of USA.

Let us recall the definition of Riemann integral. Riemann's definition in 1867 can be summarized as follows:

$$\int_a^b f(t)\,dt := \lim \sum f(t_i)(x_i - x_{i-1}).$$

Given a bounded function $f\colon [a,b] \to \mathbb{R}$, a more precise definition is as follows: we say that f is Riemann integrable on $[a,b]$ if there exists a real number α such that for a given $\varepsilon > 0$, we can find $\delta > 0$ with the property that for any partition

$$P : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b, \text{ with } \max\{x_i - x_{i_1}\} < \delta$$

we have,

$$\left| \alpha - \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) \right| < \varepsilon, \text{ for any choice of } t_i \in [x_{i-1}, x_i].$$

The Henstock-Kurzweil integral is a straightforward generalization of this.

First a couple of definitions. By a *tagged partition* of $[a,b]$, we mean a partition P as above with $t_i \in [x_{i-1}, x_i]$ chosen. The points t_i are called the tags. A positive function $\delta\colon [a,b] \to \mathbb{R}$ is called a *gauge*. We say that a tagged partition is *δ-fine* if $x_i - x_{i-1} < \delta(t_i)$ for all i. Now we are ready to define HK-integrability.

Definition 1. A function $f\colon [a,b] \to \mathbb{R}$ is said to be HK-integrable if there exists a real number α with the following property:

> For a given $\varepsilon > 0$ there exists a gauge δ such that for any δ-fine partition of $[a,b]$, we have
>
> $$\left| \alpha - \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) \right| < \varepsilon.$$

The real number α, if it exists is unique and is called the HK-integral of f and let us denote it by HK-$\int_a^b f(t)\,dt$.

Note that there is no assumption on the boundedness of f. The Riemann integrability insists on finding a constant δ in the above formulation along with the boundedness of f. What makes the HK-integral powerful is the relaxation of δ to be a positive function. In fact, in application, we select $\delta(t)$ to be extra small at points t near which the function f does not behave well. See examples below as well as the proof of Theorem 4. In geometric terms, while the Riemann integral demands that approximating rectangles have width smaller than a fixed constant width, HK-integral allows us to choose thinner rectangles near troublesome points.

For the definition to work, given a gauge δ, we need to assure the existence of a δ-fine partition. This follows by an easy application of the nested interval theorem. For, if the interval $[a, b]$ admits no δ-fine partition, neither does any of its bisected subintervals $[a, (a+b)/2]$ and $[(a+b)/2, b]$. Continuing this way, we produce a sequence of nested intervals J_n such that $\ell(J_n) = 2^{-n}(b-a)$ none which admit a δ-fine partition. Let $c \in \cap J_n$. Then for sufficiently large n, $\ell(J_n) < \delta(c)/2$. For such n, the interval J_n itself is a δ-fine partition, contradicting our choice of J_n's.

Example 2. Let f be the characteristic function of the rationals in $[0, 1]$. We claim that f is HK-integrable with integral 0. Let $\varepsilon > 0$ be given. Let $\{r_n\}$ be an enumeration of the rationals in $[0, 1]$. We define $\delta(t) = 1$ if $t \notin \mathbb{Q}$ and $\delta(r_n) = 2^{-n-1}\varepsilon$. Then in the HK-sum of any δ-fine tagged partition $\sum_{k=1}^n f(t_k)(x_k - x_{k-1})$, the only nonzero terms occur when the tag $t_k = r_n$ for some n. It is possible that the tag may be common to two adjacent subintervals. Hence $|\sum_{k=1}^n f(t_k)(x_k - x_{k-1})| < \sum_{k=1}^\infty \frac{2}{2^{n+1}}\varepsilon = \varepsilon$. Hence the claim.

Example 3. Let $f\colon [0, 1] \to \mathbb{R}$ be defined as $f(1/n) = n$ and $f(t) = 0$ at other points. We claim that $HK - \int_0^1 f(t)\,dt = 0$. Given $\varepsilon > 0$, we define $\delta(t) = \varepsilon/2^{n+1}n$ and $\delta(t) = 1$ at other points. Reasoning as above, we see that, for any δ-fine partition,

$$\left|\sum_{k=1}^{n} f(t)(x_i - x_{i-1})\right| < \varepsilon.$$

Note that this function is unbounded.

It is easily seen that if a function is Riemann integrable, then it is HK-integrable and both integrals are the same. What is more is the following: f is Lebesgue integrable iff both f and $|f|$ are HK-integrable and in such a case the Lebesgue integral of f coincides with the HK-integral of f.

We conclude this article with a very satisfactory fundamental theorem of calculus.

Theorem 4. *Let $f\colon [a, b] \to \mathbb{R}$ be continuous and differentiable on $[a, b]$. Then f' is HK-integrable on $[a, b]$ and we have HK-$\int_a^b f'(t)\,dt = f(b) - f(a)$.*

Proof. Let $\varepsilon > 0$ be given. By the differentiability of f at $t \in [a, b]$, there exists $\delta(t) >$ such that for all $x \in [a, b]$ with $|x - t| < \delta(t)$, we have

$$\left|\frac{f(x) - f(t)}{x - t} - f'(t)\right| < \varepsilon.$$

This can be rewritten as

$$\left|f(x) - f(t) - f'(t)(x-t)\right| < \varepsilon\,|x-t|\,. \tag{1}$$

Now if $x < t < y$ and $t - x < \delta(t)$ and $y - t < \delta(t)$, then we have

$$\begin{aligned} \left|f(y) - f(t) - f'(t)(y-t)\right| &< \varepsilon(y-t) \quad (2)\\ \left|f(t) - f(x) - f'(t)(t-x)\right| &< \varepsilon(t-x). \quad (3) \end{aligned}$$

Form these equations it follows, for x, t, y as above, that

$$\left|f(y) - f(x) - f'(t)(y-x)\right| < 2\varepsilon(y-x). \tag{4}$$

We take the above δ as a gauge. Let us consider any δ-fine partition. Observe that the telescoping sum

$$f(b) - f(a) = \sum_{k=1}^{n} \left(f(x_k) - f(x_{k-1})\right).$$

Hence we have

$$\left|f(b) - f(a) - \sum_{k=1}^{n} f'(t_k)(x_k - x_{k-1})\right| = \left|\sum_{k=1}^{n}(f(x_k) - f(x_{k-1}) - f'(t_k)(x_k - x_{k-1}))\right|.$$

The right hand sum is, by the triangle inequality and by (4) is less than or equal to $2\varepsilon(b-a)$. The result follows. □

I hope that your appetite for Henstock-Kurzweil integral is whetted. For further details, consult references.

REFERENCES

1. R. Bartle and Sherbert, *Introduction to Real Analysis*, 3rd ed., Wiley International.
2. R. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, Amer. Math. Soc., 1994