Any compact hypersurface in $\mathbb{R}^n$ is a level surface
(1) Orientability of Hypersurfaces in $\mathbb{R}^n$
(2) Jordan-Brouwer Separation Theorem

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Abstract

In this note we give a simple, elementary short proof of the Jordan-Brouwer Separation theorem and the orientability of hypersurface in $\mathbb{R}^n$. In fact, both are easy consequences of the main result which says that any compact hypersurface is a level set of a smooth function defined on all of $\mathbb{R}^n$.

In this note we shall prove the following result.

Theorem 1. *Let S be a compact hypersurface in $\mathbb{R}^n$. Then there exists a smooth function $f: \mathbb{R}^n \to \mathbb{R}$ such that S is a level set, i.e., $S = f^{-1}(0)$ with $\nabla f(x) \neq 0$ for all $x \in S$.*

This global result has two well-known results as corollaries. Before stating them, we recall the definition of orientability of a hypersurface in $\mathbb{R}^n$.

Let $S \subseteq \mathbb{R}^n$ be a hypersurface. S is said to be *orientable* iff there exists a continuous nowhere vanishing normal field on S. For example let $f: \mathbb{R}^n \to \mathbb{R}$ be smooth with 0 as a regular value of the image and $S = f^{-1}(0)$, then S is orientable. For, we have $x \mapsto \nabla f(x)$ is a nowhere vanishing continuous normal field on S. Here $\nabla f(x)$ denotes the gradient of f at x.

Theorem 2. *Any compact hypersurface in $\mathbb{R}^n$ is orientable.*

Proof. This is immediate from the main theorem and the characterization of orientability for a hypersurface. □

Theorem 3. [Jordan - Brouwer separation theorem] *Let $S \subseteq \mathbb{R}^n$ be a compact connected hypersurface. Then $\mathbb{R}^n \setminus S$ has only two connected components.*

Note that the we have not assumed S to be orientable in the above result.

The proof of Theorem 1 depends on three elementary lemmas, the third being the tubular neighbourhood theorem. We say that two functions $f, g: X \to \mathbb{R}$ *agree locally up to sign* if for every point $x \in X$, there exists a neighbourhood U such that either $f(y) = g(y)$ for all $y \in U$ or $f(y) = -g(y)$ for all $y \in U$. We write this as $f = \pm g$ on U.

Lemma 4. *Let X be any connected space. Let $f,\ g\colon X \to \mathbb{R}$ be maps (not necessarily continuous) that agree locally up to sign and such that the interior of $f^{-1}(0)$ and that of $g^{-1}(0)$ are empty. Then $f = g$ or $f = -g$ on X.*

Proof. Let $E = \{x \in X : f(x) = g(x)\}$. Let x be in the interior U of the closure of $\overline{E}$. Let V be any open set such that $x \in V$ and $f = \pm g$ on V. If $f(z) = 0$ for all $z \in U \cap V$, then $f^{-1}(0)$ has nonempty interior. Thus there exists $z \in U \cap V$ such that $f(z) \neq 0$. Hence $f(y) = g(y)$ for all $y \in V$ so that $V \subset U$. Hence the open set U is also closed. Since X is connected, either $U = \emptyset$ or $U = X$. Repeating the above argument with $-g$ in place of g, we see that the interior of $F := \{x \in X : f(x) = -g(x)\}$ is either empty or equal to X. Since $X = E \cup F$, the result follows. □

We need the following exercise in the next lemma.

Ex. 5. Let $K \subset \mathbb{R}^n$ be compact and $A \subset \mathbb{R}^n$ be closed. Assume further that $K \cap A = \emptyset$. Then there exist $x \in K$ and $a \in A$ such that $d(K, A) = d(x, a)$. In particular, $d(K, A) > 0$. (You may need Bolzano-Weierstrass.)

Lemma 6. *Let $\{U_\alpha\}$ be an open covering of $\mathbb{R}^n$ and $f_\alpha\colon U_\alpha \to \mathbb{R}$ be smooth maps such that (1) the interior of $f_\alpha^{-1}(0) = \emptyset$ and (2) if $U_\alpha \cap U_\beta \neq \emptyset$ then f_α and f_β agree locally up to sign on $U_\alpha \cap U_\beta$. Then there exists a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $f = \pm f_\alpha$ on U_α.*

Proof. We let U_0 denote a member of $\{U_\alpha\}$ with $0 \in U_0$. We let B_0 denote an open ball centred at 0 such that $B_0 \subset U_0$. We fix B_0 in the ensuing discussion and the corresponding f_0. Let $x \in \mathbb{R}^n$. Consider the line segment $[0, x] := \{tx : 0 \leq t \leq 1\}$ joining 0 and x. Let $[0, x] \subset \Omega := B_0 \cup B_1 \cup \cdots \cup B_r$ be a covering by means of open balls such that each B_i is a subset of some U_{α_i} of the given covering. We may arrange the matters in such a way that the following conditions are met: (i) There are no superfluous balls, i.e. each B_i meets $[0, x]$. (ii) Successive balls do intersect, i.e. $B_{i-1} \cap B_i \neq \emptyset$. (iii) Only successive balls intersect, i.e. if $B_i \cap B_j \neq \emptyset$, then i and j are consecutive.

We recursively choose f_i as follows. f_{α_1}, the function corresponding to $U_{\alpha_1} \supset B_1$ agrees locally up to sign on the connected set $B_0 \cap B_1$. We choose f_1 to be $+f_{\alpha_1}$ (resp. $-f_{\alpha_1}$), the sign being chosen so that f_1 agrees with f_0 on the connected set $B_0 \cap B_1$. By induction we choose $f_i\,|_{B_i} = f_{\alpha_i}\,|_{U_{\alpha_i}}$, the sign being chosen so that f_i agrees with f_{i-1} on the connected set $B_{i-1} \cap B_i$. We define $f(x) := f_r(x)$, the last of the f_i's. We need to show that $f(x)$ is well-defined.

Suppose $[0, x] \subset \Omega' := B_0 \cup B_1' \cup \cdots \cup B_s'$. (Note that the 0-th ball is fixed always.) We denote by f_j' the functions chosen as above. We need to show that $f_r(x) = f_s'(x)$. Note that the set $\Omega \cap \Omega' := (B_0 \cup B_1 \cup \cdots \cup B_r) \cap (B_0 \cup B_1' \cup \cdots \cup B_s')$ is connected. By Lemma 4, $f_r = f_s'$ on this set, since $f_0 = f_0'$. In particular, $f_r(x) = f_s'(x)$ so that $f(x)$ is well-defined.

We now show that f is smooth. Note that $[0, x]$ is compact, $\mathbb{R}^n \setminus \Omega$ is closed and they are disjoint subsets of $\mathbb{R}^n$. Hence $d([0, x], \mathbb{R}^n \setminus \Omega) > 2\varepsilon > 0$, by Exercise 1. If $y \in B(x, \varepsilon)$ and $\Omega'' := B_0 \cup B_1'' \cup \cdots B_t'' \supset [0, y]$, where each B_i' has radius less than ε and centre on $[0, y]$, then $\Omega'' \subset \Omega$. The corresponding function f_t'' equals $f_r\,|_{\Omega''}$ by Lemma 4. Consequently, $f(y) = f_t''(y) = f_r(y)$. Hence f agrees with f_r on $B(x, \varepsilon)$ and hence is smooth. □

The following lemma is a version of tubular neighbourhood theorem and is stated in this form as we do not assume the orientability of S.

Lemma 7 (Tubular Neighborhood Theorem)**.** *Let S be a compact connected (not necessarily orientable) hypersurface in $\mathbb{R}^n$. Given δ, we denote by $U_\delta(S)$ (respectively, $U_\delta[S]$) the union of all open (respectively, closed) line segments of length 2δ that are normal to the surface S and centred at a point of S. There exists an ε such that any two such line segments with distinct centres are disjoint. The set $U_{2\varepsilon}(S)$ is called an ε-tubular neighbourhood of S.*

Proof. Recall that if S is a hypersurface and $x \in S$, there is a neighbourhood U_x on which there is a smooth unit normal field N_x. If we further assume that U_x is connected then there are exactly two such smooth unit normal fields, each being the negative of the other. We fix one of them and denote it by N_x or N if there is no possible source of confusion. We consider the map $\varphi := \varphi_x \colon U_x \times \mathbb{R} \to \mathbb{R}^n$ given by $\varphi(z) := z + tN(z)$, for $z \in U_x$. We claim that φ is a local diffeomorphism at $(x, 0)$. To see this, we need to show that the derivative $D\varphi(x, 0) \colon T_xS \times \mathbb{R} \to \mathbb{R}^n$ is nonsingular. We achieve this by showing that the said derivative maps a basis of the domain onto a basis of the range. Let $v \in T_xS$. We choose a smooth curve $c \colon (-\varepsilon, \varepsilon) \to S$ such that $c(0) = x$ and $c'(0) = v$. Let $\gamma(s) := (c(s), 0) \in S \times \mathbb{R}$. Then

$$D\varphi(x,0)(v,0) = \frac{d}{ds}\left(\varphi \circ \gamma(s)\right)|_{s=0} = \frac{d}{ds}((c(s),0))|_{s=0} = (c'(0),0) = (v,0).$$

Also, if we let $\sigma(s) := (x, s)$ then

$$D\varphi(x,0)(0,1) = \frac{d}{ds}\left(\varphi \circ \sigma(s)\right)|_{s=0} = (0,1).$$

These computations shows that the derivative $D\varphi(x, 0)$ is nonsingular. Hence φ maps a neighbourhood of $(x, 0)$ diffeomorphically onto a neighbourhood of x in $\mathbb{R}^n$. In particular, there exists a neighbourhood U_x of x in S and an $\varepsilon_x > 0$ such that φ is one-to-one on $U_x \times (-\varepsilon_x, \varepsilon_x)$.

After this differential argument, we complete the proof by a compactness argument. Let us suppose that there exists no ε as stated in the theorem. Then there exists a sequence of distinct pairs (x_k, s_k) and (y_k, t_k) in $S \times (-1/k, +1/k)$ such that $x_k + s_kN(x_k) = y_k + t_kN(y_k)$ for all $k \in \mathbb{N}$. By compactness, after passing to subsequences (possibly twice), we may assume that $x_k \to x$ and $y_k \to y$ in S. We claim that $x = y$:

$$x = \varphi(x,0) = \lim \varphi(x_k, s_k) = \lim \varphi(y_k, t_k) = \varphi(y,0) = y.$$

Fix a neighbourhood $U_x \times (-\varepsilon_x, \varepsilon_x)$ of $(x, 0)$ on which φ is one-to-one. Then, for all sufficiently large k, (x_k, s_k) and (y_k, t_k) lie in this neighbourhood. This contradicts the one-oneness of φ on U_x. Whence we conclude that there exists an ε as required. □

Proof of Theorem 1. Let ε be such that $U_{2\varepsilon}(S)$ is a 2ε-tubular neighbourhood of S. Observe that $U_\varepsilon[S]$ is a closed subset of $\mathbb{R}^n$. Fix a smooth function $\alpha \colon \mathbb{R} \to \mathbb{R}$ such that $\alpha(0) = 0$, $\alpha'(0) = 1$, $\alpha'(t) > 0$ for $0 \le t < \varepsilon$, $\alpha(t) = \varepsilon$ for $t \ge \varepsilon$ and finally $\alpha(-t) = -\alpha(t)$. We wish to apply Lemma 6. First of all, we cover S with a family of neighbourhoods $\{V_i\}$ on which a smooth unit normal field u_i is defined. We fatten members of this family to get open sets in $\mathbb{R}^n$: Let $U_i := \{x + tu_i(x) : x \in V_i, |t| < 2\varepsilon\}$. This family $\{U_i\}$ along with $U^* := \mathbb{R}^n \setminus U_\varepsilon[S]$ forms an open cover of $\mathbb{R}^n$. On U_i we define $f_i(x + tu_i(x)) := \alpha(t)$. Also, $f_*: U^* \to \mathbb{R}$ is defined to be the constant ε. We easily check that the hypothesis of Lemma 6 are satisfied so that we obtain a smooth function $f: \mathbb{R}^n \to \mathbb{R}$ with $f^{-1}(0) = S$. Also, a simple calculation shows that $\operatorname{grad} f(x) := \pm u(x) \neq 0$ for all $x \in S$. □

Theorem 3 is an immediate consequence of Theorem 1 and the following lemma.

Lemma 8. *Let* $f: \mathbb{R}^n \to \mathbb{R}$ *be smooth. Let* $S := f^{-1}(0) \neq \emptyset$. *Assume that* $\operatorname{grad} f(x) \neq 0$ *for* $x \in S$. *Assume further that* S *is compact and connected. Then* $\mathbb{R}^n \setminus S$ *has two connected components and* S *is their common boundary.*

Proof. Let $A := \{x \in \mathbb{R}^n : f(x) > 0\}$ and $B := \{x \in \mathbb{R}^n : f(x) < 0\}$. Then A and B are open, disjoint and $\mathbb{R}^n \setminus S = A \cup B$. We claim that A and B are connected. Observe that A contains the connected closed subset $C := \{x + t \operatorname{grad} f(x) : x \in S, 0 < t < 2\varepsilon\}$. Let $y \in A$. Let p be point on $U_\varepsilon[S]$ such that $d(y, U_\varepsilon[S]) = d(y, p)$. (Such a p exists in view of Exercise 1.) Note that the line segment $[p, y] \subset A$. Thus any point $y \in A$ either lies in C or can be joined to a point on C by means of a line segment. Consequently, A is connected. Similarly, one shows that B is connected.

Letting ∂E denote the (topological) boundary of E, we observe that $f(\partial A) \subset S$, as f must vanish on ∂A. Similarly, $\partial B \subset S$. Since any neighbourhood (in $\mathbb{R}^n$) of a point $x \in S$ meets both A and B, we see that $S \subset \partial A \cap \partial B$. Hence $S = \partial A = \partial B$. □