Implicit Function Theorem

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The aim of this article is bring out the geometric content of the implicit function theorem of two variables. To make it self-contained, I have included a proof of the theorem. The exposition is aimed at students of mathematics from final year B.Sc. or first year M.Sc.

We shall look at three examples before stating the theorem.

Let us look at the function $f: \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) := ax + by$. Assume that $(a, b) \neq (0, 0)$. We look at the set $S_c := \{(x, y) \in \mathbb{R}^2 : f(x, y) = c\} \equiv f^{-1}(c)$. This set S_c is called the level set of f at the level c. This is a line in $\mathbb{R}^2$. Intuitively, S_c is "one dimensional", i.e., to locate a point $(x, y) \in S_c$, we need only one of the coordinates. For instance, if we assume that $b \neq 0$ and if the x coordinate of a point in S_c is given, then its y-coordinate is given by $y = (c - ax)/b$. Thus, we have a function $g: \mathbb{R} \to \mathbb{R}$ by setting $g(x) = (c - ax)/b$ for all $x \in \mathbb{R}$ so that $f(x, g(x)) = c$. In this case, it turns out that the level set is just the graph of the function $g: \mathbb{R} \to \mathbb{R}$.

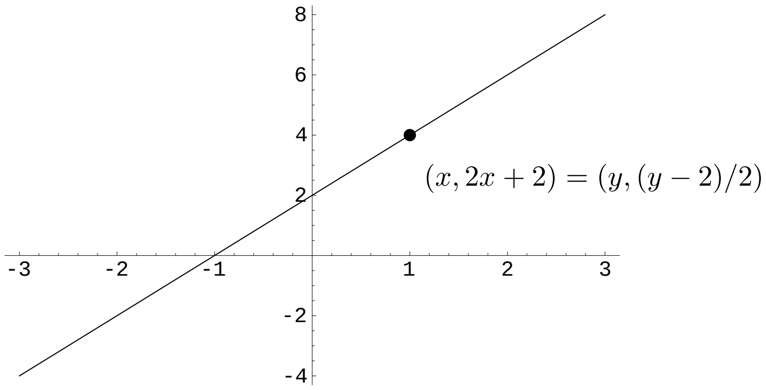

Figure 1: The line $2x - y + 2 = 0$ as the graph of $f(x) = 2x + 2$ or of $g(y) = (y - 2)/2$

Let us look at another example. Consider $f: \mathbb{R}^2 \to \mathbb{R}$ where $f(x, y) = x^2 + y^2$. For $c = 1$, the level set S_1 contains the point $(0, 1)$. Can we find a single function φ on $[-1, 1]$ so that S_1 is the graph of φ? An obvious choice is $\varphi(x) := \sqrt{1 - x^2}$, where the square root is taken to be the positive (or the negative) one for all x. Thus, if we want our resulting φ to be a function, then we need to restrict ourselves to a part of S_1, viz., $S_1^+ := \{(x, y) \in S_1 : y \geq 0\}$ (or S_1^- defined analogously). In particular, we cannot exhibit S_1 as a graph of a single function.

However, note that, we can write S_1 as a union of these two sets $S_1^{\pm}$ which by themselves are graphs. But if we insist that the domains of these functions φ be open intervals, we then need to write S_1 as a union of four overlapping pieces. (Exercise: Do this. Look at Figures 2 and 3.)

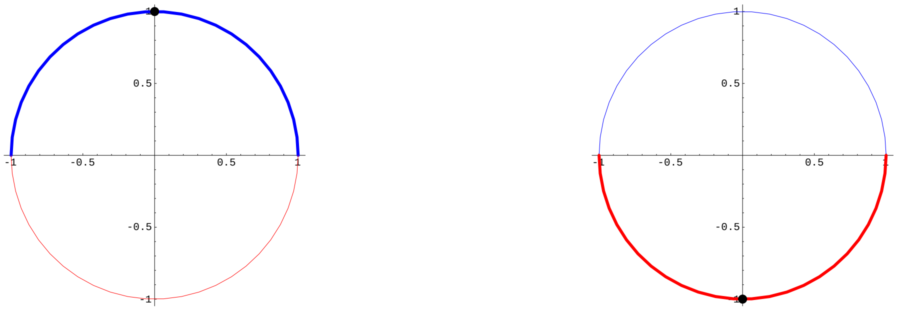

Figure 2: The thick arcs are the graphs of $x \mapsto \pm\sqrt{1-x^2}$

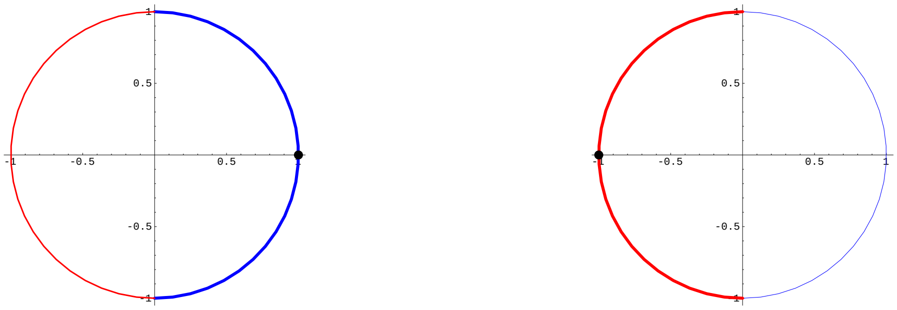

Figure 3: The thick arcs are the graphs of $y \mapsto \pm\sqrt{1-y^2}$

As the third example, let us consider $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = xy$ and the level set S_0 of f. We ask you to convince yourself that it is not possible to exhibit S_0 in **any** neighbourhood around the origin as the graph of a function defined on an interval around $0 \in \mathbb{R}$.

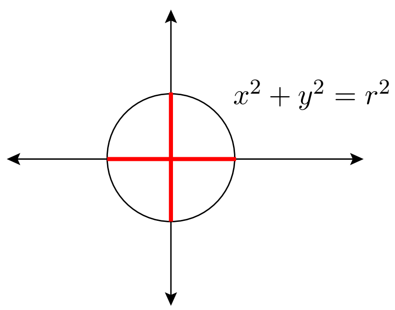

Figure 4: Can the thick line segments be a graph?

The implicit function theorem tells us under certain conditions, we can exhibit level sets as graphs "locally". What does this word "locally" mean? It means that given a point (x_0, y_0)

in the level set, there exists an open ball centred at (x_0, y_0) in $\mathbb{R}^2$ such that portion of the level set in this ball will be the graph of a suitably defined function. Loosely speaking, for all points near to (x_0, y_0) we can express one of the coordinates as a function of the other coordinate so that the level set around this point is "one dimensional".

With these basic ideas, we can state the precise version of the theorem.

Theorem 1 (Implicit Function Theorem)**.** *Let $U \subset \mathbb{R}^2$ be open and $f: U \to \mathbb{R}$ have continuous partial derivatives, f_x and f_y on U. Assume that $(x_0, y_0) \in U$ is such that $f(x_0, y_0) = 0$ and that $f_y(x_0, y_0) \neq 0$. Then there exist an $\varepsilon > 0$ and a function $g: (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$ such that the following hold:* (1) *$f(x, g(x)) = 0$ for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$ and* (2) *g is differentiable on its domain with*

$$g'(x) = \frac{-f_x(x, g(x))}{f_y(x, g(x))}.$$

The proof depends decisively on the mean value theorem which we recall in the form which we need.

Theorem 2 (Mean Value Theorem)**.** *Let $f: U \subset \mathbb{R}^2 \to \mathbb{R}$ be differentiable. Let us assume that $p = (a, b)$ and $z = (x, y)$ be points of U such that the line segment joining them lies completely in U. Then there exists a point (x_1, y_1) on this line segment such that*

$$f(z) - f(p) = f_x(x_1, y_1)(x - a) + f_y(x_1, y_1)(y - b).$$

Proof. Consider the one variable function $g(t) := f(p + t(z - p))$ for $t \in [0, 1]$. Apply the one variable mean value theorem and the chain rule to get the result. □

We also need the following fact.

Theorem 3. *Let $K \subset \mathbb{R}^2$ be closed and bounded and $f: K \to \mathbb{R}$ be continuous. Then*
(1) *f is bounded, i.e., there exists a constant $A > 0$ such that $|f(x)| \leq A$ for all $x \in K$.*
(2) *If we further assume that $f(x) > 0$ for all $x \in K$, then there exists $B > 0$ such that $f(x) \geq B$ for all $x \in K$.*

Proof. We use the Bolzano-Weierstrass theorem: *Any bounded sequence in $\mathbb{R}^2$ has a convergent subsequence.* (See the exercise below for a hint.)

If f is not bounded, then there exists a sequence of points $z_n \in K$ such that $f(z_n) > n$ for $n \in \mathbb{N}$. Let (z_{n_k}) be a convergent sequence converging to $z \in \mathbb{R}^2$. Since K is closed, $z \in K$. By continuity of f, we have $f(z_{n_k}) \to f(z)$. For $\varepsilon = 1$, by continuity of f, there exists a $\delta > 0$ such that if $w \in B(z, \delta)$, then $|f(w) - f(z)| < 1$. This implies that $|f(z_{n_k})| < |f(z)| + 1$ for all k sufficiently large. This contradiction proves that there exists $N \in \mathbb{N}$ such that $|f(z)| \leq N$ for all $z \in K$.

To prove 2), assume that there exists no $n \in \mathbb{N}$ such that $f(z) \geq 1/n$ for all $z \in K$. Then there exists a sequence (z_n) in K such that $f(z_n) < 1/n$ for $n \in \mathbb{N}$. Arguing as in the first part, we see that $f(z) = 0$ for some $z \in K$. □

Ex. Assume the Bolzano-Weierstrass theorem in $\mathbb{R}$. If (z_n) is a bounded sequence in $\mathbb{R}^2$, where $z_n := (x_n, y_n)$, then (x_n) and (y_n) are bounded sequences in $\mathbb{R}$. Let (x_{n_k}) be convergent to x in $\mathbb{R}$. Apply Bolzano-Weierstrass theorem in $\mathbb{R}$ to the bounded sequence (y_{n_k}). It has a convergent sequence, say, $(y_{n_{k_r}})$ converging to y. Then $(z_{n_{k_r}})$ is convergent to $z = (x, y)$.

In case you know the concept of compact sets in topology or metric spaces, we prove a more general statement which includes the above as a special case. (You still need Heine-Borel theorem which asserts that a set in $\mathbb{R}^2$ is compact iff it is closed and bounded.)

Ex. 4. Let $a, b, c, d \in \mathbb{R}$ be such that $b-a = d-c$. Let $S := [a, b] \times [c, d]$ be the square in $\mathbb{R}^2$. The vertices of S are (a, c), (b, c), (b, d) and (a, d). We call the point (a, c) as the bottom left vertex of S. The pair of midpoints of its opposite sides are given by $([a+b]/2, c), ([a+b]/2, d)$ and $(a, [c+d]/2), (b, [c+d]/2])$. By joining the midpoints of opposite sides, we get four smaller squares. Observe that if (a_1, c_1) is the bottom left vertex of any of these squares, we have $a \leq a_1$ and $c \leq c_1$.

Theorem 5. *A subset of $\mathbb{R}^2$ is compact iff it is closed and bounded.*

Proof. Let K be a closed and bounded set in $\mathbb{R}^2$. Then there exists $R > 0$ such that $K \subset S := [-R, R] \times [-R, R]$. Since a closed subset of a compact set is compact, it suffices to show that S is compact.

Suppose that S is not compact. Then there is an open cover $\{U_i : i \in I\}$ of which there is no finite subcover of S. Let us divide the square S into four smaller squares by joining the pairs of midpoints of opposite sides. (See Exercise 4 above.) One of these square will not have a finite subcover from the given cover. For, otherwise, all these four squares will have finite subcovers so that S itself will admit a finite subcover. Choose one such smaller square and call it S_1. Note that the length of its sides is R and that if (a_1, b_1) is the bottom left vertex of S_1, then $a_1 \geq a_0 = -R$ and $c_1 \geq c_0 = -R$. We repeat the argument by subdividing S_1 into four squares and choosing one of the smaller squares which does not admit a finite subcover of $\{U_i\}$. Call this smaller square as S_2. Note that the length of its sides is $R/2$ and that if (a_2, c_2) is the bottom left vertex of S_2, then $a_1 \leq a_2$ and $c_1 \leq c_2$.

Proceeding recursively, we have a sequence of squares S_n such that S_n dose not admit a finite subcover and the length of sides of S_n is $2^{-n+1}R$ and its bottom left vertex (a_n, c_n) is such that $a_{n-1} \leq a_n$ and $c_{n-1} \leq c_n$. Thus we have two sequences of real numbers (a_n) and (c_n). They are bounded and monotone. Hence there exist real numbers a and c such that $a_n \to a$ and $b_n \to b$. It follows that $(a_n, c_n) \to (a, c) \in \mathbb{R}^2$. Since S is closed, we infer that $(a, b) \in S$. Hence there is U_{i_0} in the open cover such that $(a, c) \in U_{i_0}$. Since U_{i_0} is open there exists an $r > 0$ such that $B((a, c), r) \subset U_{i_0}$.

Choose $n \in \mathbb{N}$ so that (1) $\operatorname{diam} S_n = 2^{-n+1}\sqrt{2}R < r/2$ and (2) $d((a, c), (a_n, c_n)) < r/2$. We then have, for any $(x, y) \in S_n$,

$$d((a, c), (x, y)) \leq d((a, c), (a_n, c_n)) + d((a_n, c_n), (x, y)) < r/2 + 2^{-n+1}\sqrt{2}R < r.$$

Thus $S_n \subset B((a, c), r) \subset U_{i_0}$. But then $\{U_{i_0}\}$ is a finite subcover for S_n, contradicting our choice of S_k's. Therefore, or assumption that S is not compact is not tenable. $\square$

Theorem 6 (Heine-Borel)**.** *A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.*

Proof. One can adapt the proof of Thm. 5 to prove the theorem including the case when $n = 1$. We leave the details to the reader. $\square$

Theorem 7. *Let $f\colon X \to \mathbb{R}$ be a continuous function from a compact (metric) space to $\mathbb{R}$. Then*
(1) f is bounded, i.e., there exists a constant $A > 0$ such that $|f(x)| \leq A$ for all $x \in X$.
(2) If we further assume that $f(x) > 0$ for all $x \in X$, then there exists $B > 0$ such that $f(x) \geq B$ for all $x \in X$.

Proof. To prove (1), consider the open sets $U_n := \{x \in X : |f(x)| < n\}$ for $n \in \mathbb{N}$. Then $U_n \subset U_{n+1}$ and $X = \cup U_n$.

To prove (2), consider $V_n := \{x \in X : f(x) > 1/n\}$. □

We now start with the proof of the main theorem. We may assume, without loss of generality, that $f_y(x_0, y_0) > 0$. By continuity of f_y in U, there exists a $\delta > 0$ such that $f_y(x,y) > 0$ in the closed square $Q := [x_0-\delta, x_0+\delta]\times[y_0-\delta, y_0+\delta]$. Either by Theorem 3 or by Theorem 7, there exist positive constants A and B such that $|f_x(x,y)| \leq A$ and $f_y(x,y) \geq B$ for all $(x,y) \in Q$. We want to choose an $\varepsilon > 0$ such that f is negative (respectively positive) on the bottom (resp. top) side of the rectangle $R := [x_0 - \varepsilon, x_0 + \varepsilon] \times [y_0 - \delta, y_0 + \delta]$.

Using the mean value theorem, we get the following:

$$\begin{aligned} f(x, y_0 - \delta) - f(x_0, y_0) &= f_x(x_1, y_1)(x - x_0) + f_y(x_1, y_1)(-\delta) \\ &\leq A\varepsilon - B\delta \end{aligned}$$

which is negative, provided that $\varepsilon < \min\{\delta, (B\delta)/A\}$. Similarly, we see that $f(x, y_0 + \delta) > 0$ provided that $\varepsilon < \min\{\delta, (B\delta)/A\}$.

Fix any $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$. Now if we consider the function of one variable $y \to f(x,y)$ on the interval $[y_0 - \delta, y_0 + \delta]$, the function assumes values of opposite signs at the end points of the interval. Hence there exists at least one y in this interval such that $f(x,y) = 0$. But the derivative of this function of y is $f_y(x,y) > 0$ and hence is strictly increasing function of y in this interval. Hence there exists a unique $y \in [y_0 - \delta, y_0 + \delta]$ such that $f(x,y) = 0$. Thus for a given $x \in [x_0 - \varepsilon, x_0 + \varepsilon]$, there exists a unique $y \in [y_0 - \delta, y_0 + \delta]$ such that $f(x,y) = 0$. We call the y that corresponds to the given x as $g(x)$. Thus we get a function $g\colon [x_0 - \varepsilon, x_0 + \varepsilon] \to \mathbb{R}$ such that $f(x, g(x)) = 0$.

We now show that g is continuous. Let x_1, x_2 be in the domain of g. Let $y_j = g(x_j)$ for $j = 1, 2$. Then, for some point (x_3, y_3) lying on the line segment joining (x_j, y_j), $j = 1, 2$, we have

$$0 = f(x_1, y_1) - f(x_2, y_2) = f_x(x_3, y_3)(x_1 - x_2) + f_y(x_3, y_3)(y_1 - y_2),$$

so that

$$g(x_1) - g(x_2) = -\frac{f_x(x_3, y_3)}{f_y(x_3, y_3)}(x_1 - x_2). \tag{1}$$

Notice that the right side makes sense since $(x_3, y_3) \in R$ and over there $f_y \neq 0$. From Eq. 1, it follows that

$$|g(x_1) - g(x_2)| \leq \frac{A}{B}|x_1 - x_2|.$$

This establishes the (Lipschitz and hence the uniform) continuity of g.

We now use Eq. 1 to show that g is differentiable at any point $x_1 \in (x_0 - \varepsilon, x_0 + \varepsilon)$:

$$\begin{aligned}
\lim_{x_2 \to x_1} \frac{g(x_1) - g(x_2)}{x_1 - x_2} &= -\lim_{x_2 \to x_1} \frac{f_x(x_3, y_3)}{f_y(x_3, y_3)} \\
&= -\frac{f_x(x_1, g(x_1))}{f_y(x_1, g(x_1))}
\end{aligned}$$

by the continuity of the partial derivatives and that of g. □

Remark 8. The above argument can be modified to prove the implicit function theorem for real valued functions of n-variables where $n \geq 2$. We leave the formulation and its proof to the interested readers.

We now wish to bring out the geometric meaning of the condition that $f_y(x_0, y_0) \neq 0$. First of all note that if $f_x(x_0, y_0) \neq 0$, we could have proceeded as above and exhibited x as a function of y. Thus, the correct condition is that $(f_x(x_0, y_0), f_y(x_0, y_0)) \neq (0, 0)$. That is, the gradient $\nabla f(x_0, y_0)$ of f at (x_0, y_0) is nonzero. So, the question is: what is the geometric meaning of $\nabla f(x, y)$?

We again look at some examples to develop our geometric intuition. Consider a nonzero linear function $f(x, y) = ax + by$, for $(x, y) \in \mathbb{R}^2$. The level sets are lines parallel to the line $ax + by = 0$ and the gradient of f at any point can be thought of as the normal to the line at the point.

As a second example, consider $g(x, y) := x^2 + y^2$. Then the level sets S_c for $c > 0$ are concentric circles with origin as the centre and radius $\sqrt{c}$. The gradient of g at (x, y) is $2(x, y)$ which is normal to the level set $S_{\sqrt{x^2+y^2}}$ **at** the point (x, y). The reader may like to look at another example such as $h(x, y) = xy$ or more generally $h(x, y) = ax^2 + 2hxy + by^2$. Using his knowledge of two dimensional coordinate geometry, he can convince himself that the gradient $\nabla h(x, y)$ of h at $(x, y) \in S_c$ is normal to the "conic" S_c **at** the point (x, y).

Therefore, the geometric meaning of the gradient condition of the implicit function theorem is that the level set S_0 has a nonzero normal at (x_0, y_0). Put in a different way, the condition assures that the "one dimensional object" or the "curve " S_0 has a well-defined tangent line at (x_0, y_0). To understand this we invite the reader to examine the third example (in the beginning) more closely. At the origin the gradient is zero and the "curve" has no tangent line. But, however, observe that away from the origin, we can express exactly one of the coordinate as the constant function of the other and that near those points the curve has a normal and hence a tangent line. (Verify these claims.)

One last remark. If we go through the proof of the implicit function theorem, we realize what we have achieved: Under the stated conditions, there exists a neighbourhood V of (x_0, y_0) in U such that we have a continuous bijection $\psi\colon (x_0 - \varepsilon, x_0 + \varepsilon) \to V \cap S_0$. The inverse ψ^{-1}, being the projection onto the first coordinate, is obviously continuous and hence ψ is a *homeomorphism* of the interval onto the portion of S_0 inside V. This is the reason why we kept saying that the implicit function theorem exhibits level sets of f, (satisfying certain conditions), as one dimensional objects. Equivalently, if you think of S_0 as a "curve" defined implicitly, then the implicit function theorem gives a local parametrization of S_0 around (x_0, y_0), viz., $x \mapsto (x, g(x))$ for $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$.

Going through this article a couple of times and trying your level best to have an intuitive feel could be a stepping stone to modern differential geometry.

Optional: Lagrange Multiplier Method

Theorem 9 (Lagrange Multiplier)**.** *Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ and $g\colon U \subset \mathbb{R}^n \to \mathbb{R}$ be C^1. Let $p \in U$, $g(p) = \alpha$ and $S := g^{-1}(\alpha)$, the level set of g at α. Assume that $\nabla g(p) \neq 0$. If the restriction of f to S has a local extremum at p, then there is a real number λ such that $\nabla f(p) = \lambda \nabla g(p)$.*

Proof. Assume without loss of generality that $\frac{\partial g}{\partial x_n}(p) \neq 0$. By the implicit function theorem, there exists an h such that

$$g(x', h(x')) := g(x_1, \dots, x_{n-1}, h(x_1, \dots, x_{n-1})) = 0 \qquad \text{for all } x' \subset U' \subset \mathbb{R}^{n-1}.$$

Let $\varphi(x) := f(x', h(x'))$. If φ has an extremum at p, then

$$0 = \frac{\partial \varphi}{\partial x_i}(p) = \left(\frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_n}\frac{\partial h}{\partial x_i} \right)(p) \qquad 1 \leq i \leq n-1. \tag{2}$$

But $g(x', h(x')) = 0$ implies that

$$\frac{\partial g}{\partial x_i} + \frac{\partial g}{\partial x_n}\frac{\partial h}{\partial x_i}, \qquad 1 \leq i \leq n-1. \tag{3}$$

It follows from Eq. 2 and Eq. 3 that

$$\frac{\partial f}{\partial x_i} = \frac{\frac{\partial f}{\partial x_n}}{\frac{\partial g}{\partial x_n}} \frac{\partial g}{\partial x_i} \text{ at } p.$$

Hence if we take $\lambda := \frac{\frac{\partial f}{\partial x_n}}{\frac{\partial g}{\partial x_n}}(p)$, the result follows. □

To explain the geometry behind the Lagrange multiplier technique, we shall look at a simple example. We want to find the extrema of the function

$$f(x, y) = xy \text{ subject to the constraint } (x^2/9) + (y^2/4) = 1.$$

The level curves $f(x, y) =$ constant are rectangular hyperbolas (inclusive of degenerate case). To apply the theorem, we take $g(x, y) = (x^2/9) + (y^2/4) - 1$. The gradients of f and g are given by $\nabla f(x, y) = (y, x)$ and $\nabla g(x, y) = (2x/9, y/2)$. By Lagrange's theorem, we are looking for a scalar λ such that $\nabla f(x, y) = \lambda \nabla g(x, y)$. Eliminating λ, we see that the points on the ellipse at which these gradients are proportional are $(\pm\frac{3}{\sqrt{2}}, \pm\sqrt{2})$.

As you can see in Figure 5 these points on the ellipse are the points where the ellipse and the level curves (namely, the hyperbolas) have common tangent. (The gradient f is a multiple of that of g is same as saying that the normal of the ellipse at that point is a multiple of the normal of the level curve, hyperbola.) Also, observe that we have plotted the gradients at these points. The thicker arrows are the gradients of g (that is, the normals to the ellipse)

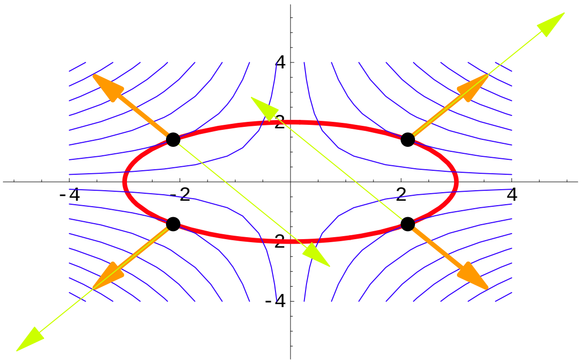

Figure 5: An Illustration of Lagrange's Method

and the thinner ones are that of f (that is, normals to the hyperbola). The direction of the thinner arrows tell us whether the point is a constrained maximum or a minimum. Can you guess why?

As a second illustration, let us find the maximum and minimum values of the function $f(x, y) = 2x + 3y$ subject to the constraint $g(x, y) = x^2 + y^2 - 1$. See Figure 6.

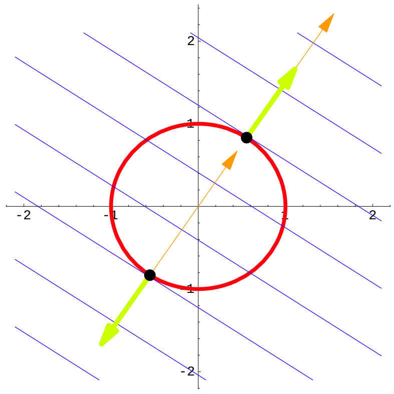

Figure 6: 2nd Illustration of Lagrange's Method

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