Applications of Baire Category Theorem

S. Kumaresan
School of Math. and Stat.
University of Hyderabad
Hyderabad 500046
kumaresa@gmail.com

1 Baire Category Theorem

Definition 1. A subset $A \subset X$ of a topological space is said to be *nowhere dense* in X, if given any nonempty open set U, we can find a nonempty open subset $V \subset U$ such that $A \cap V = \emptyset$.

This definition is equivalent to the standard one found in all text-books: A is nowhere dense in X iff the interior of the closure of A in X is empty.

I prefer the first one, as it gives us a better geometric intuition as it uses only the primitive concepts of topology.

Example 2. Let V be any proper vector subspace of $\mathbb{R}^n$. Then V is nowhere dense in $\mathbb{R}^n$. This is a typical example of a nowhere dense set.

More generally, let X be a normed linear space. Let V be any proper vector subspace of X. Then V is nowhere dense in X.

We shall give the formulation of Baire category theorem in a form which will be more useful than the one which uses the notion of category.

Theorem 3 (Baire Category Theorem)**.** *Let (X, d) be a complete metric space.*
(1) *Let U_n be open dense subsets of X, for $n \in \mathbb{N}$. Then $\cap_n U_n$ is dense in X.*
(2) *Let F_n be nonempty closed subsets of X such that $X = \cup_n F_n$. Then at least one of F_n's has nonempty interior. In other words, a complete metric space cannot be a countable union of nowhere dense closed subsets.*

Proof. We first observe that both the statements are equivalent. For, G is open and dense iff its complement $F := X \setminus G$ is closed and nowhere dense. Hence any one of them follows from the other by taking complements. So, we confine ourselves to proving the first.

Let $U := \cap_n U_n$. We have to prove that U is dense in X. Let $x \in X$ and $r > 0$ be given. We need to show that $B(x, r) \cap U \neq \emptyset$. Since U_1 is dense and $B(x, r)$ is open there exists $x_1 \in B(x, r) \cap U_1$. Since $B(x, r) \cap U_1$ is open, there exists r_1 such that $0 < r_1 < 1/2$ and $B[x_1, r_1] \subset B(x, r) \cap U_1$. We repeat this argument for the open set $B(x_1, r_1)$ and the dense set U_2 to get $x_2 \in B(x_1, r_1) \cap U_2$. Again, we can find r_2 such that $0 < r_2 < 2^{-2}$ and $B[x_2, r_2] \subset B(x_1, r_1) \cap U_2$. Proceeding this way, we get for each $n \in \mathbb{N}$, x_n and r_n with the properties

$$B[x_n, r_n] \subset B(x_{n-1}, r_{n-1}) \cap U_n \text{ and } 0 < r_n < 2^{-n}.$$

Clearly, the sequence (x_n) is Cauchy: if $m \leq n$,

$$d(x_m, x_n) \leq d(x_n, x_{n-1}) + \cdots + d(x_{m+1}, x_m) \leq \sum_{k=m}^{n} 2^{-k}.$$

Since $\sum_k 2^{-k}$ is convergent, it follows that (x_n) is Cauchy.

Since X is complete, there exists $x_0 \in X$ such that $x_n \to x_0$. Since x_0 is the limit of the sequence $(x_n)_{n\geq k}$ in the closed set $B[x_k, r_k]$, we deduce that $x_0 \in B[x_k, r_k] \subset B(x_{k-1}, r_{k-1}) \cap U_k$ for all k. In particular, $x_0 \in B(x, r) \cap U_k$. □

Remark 4. The importance of our formulation is this. The first statement tells us of a typical way in which Baire category can be used. Imagine that we are on the look-out for an element $x \in X$ with some specific properties. Further assume that the sets of elements which have properties "arbitrarily close" to the one desired are dense open sets in X. Then the result says that there exists at least one element with the desired property. Thus the first formulation is useful when we are interested in the existence problems. This vague way of remembering is well-illustrated in some of the applications below. See especially the existence of everywhere continuous nowhere differentiable function.

The second formulation says that X cannot be a countable union of "hollow" sets. A typical application: $\mathbb{R}^n$ cannot be the union of a countable collection of lower dimensional subspaces. Another instance: a complete normed linear space cannot be countable dimensional. See Prop. 8 below.

2 Uniform Boundedness Principle

Theorem 5. *Let X be a complete metric space and $\mathcal{F}$ a family of continuous real valued functions on X. Assume that for all $x \in X$, there exists $C_x > 0$ such that $|f(x)| \leq C_x$ for all $f \in \mathcal{F}$. Then there exists a non-empty open set $U \subseteq X$ and a constant C such that $|f(x)| \leq C$ for all $f \in \mathcal{F}$ and $x \in U$.*

Proof. For each $f \in \mathcal{F}$ and $n \in \mathbb{N}$, let $E_{n,f} = \{x \in X \mid |f(x)| \leq n\}$. $E_{n,f}$ is closed since f is continuous. Let $E_n = \cap_{f\in\mathcal{F}} E_{n,f}$. Then E_n is closed and $\cup E_n = X$, since if $x \in X$, $x \in E_n$ if $n > C_x$. By Baire category, not all E_n's are nowhere dense. Hence $\overline{E_n} = E_n$ must contain a non-empty open set U for some n. Take $C = n$. □

Theorem 6. . *Let X be a Banach space and C a closed, convex, symmetric subset of X such that $\cup_n nC = X$. Then C is a neighborhood of* 0.

Proof. All nC are closed since multiplication by a non-zero scalar is a homeomorphism. Hence there exists $N > 0$ such that NC has a non-empty interior, by Baire Category Theorem and hence C has a non-empty interior (multiply by $\frac{1}{N}$!). Hence there exists $x \in C$ and a

$r > 0$ such that $x + r(B(0,1)) \subseteq C$. Since C is symmetric, $-x \in C$, so that by convexity, $\frac{1}{2}r(B(0,1)) = \frac{1}{2}(-x) + \frac{1}{2}(x + B(0,1)) \subseteq C$. That is, C contains the neighborhood $B(0, \frac{r}{2})$ of 0. □

Remark 7. This underlies the proof of the open mapping theorem.

Proposition 8. *Let X be an infinite dimensional complete normed linear space. Then X cannot be countable dimensional.*

Proof. Let, if possible, $\{e_n : n \in \mathbb{N}\}$ be a (Hamel/algebraic) basis of X. This means that any vector $x \in X$ is a finite linear combination of e_n's. If we let F_n stand for the vector subspace spanned by $\{e_k | 1 \leq k \leq n\}$, then F_n is finite dimensional. It is well-known that all norms on a finite dimensional vector space are equivalent so that any finite dimensional normed linear space is necessarily complete. In view of this we conclude that each F_n is closed in X. By Eg. 2, F_n is closed for each n. By hypothesis, the complete metric space X is the union of the countable family $\{F_n\}$ of nowhere dense closed sets, a contradiction. □

Theorem 9. *If $f_n \in \mathcal{C}[a,b]$ are such that $f_n(t) \to f(t)$ for each $t \in [a,b]$, then the set of points of continuity of f is dense in $[a,b]$.*

We first prove a lemma.

Lemma 10. *To each closed subinterval I of $[a,b]$ and every $\varepsilon > 0$, there exists a closed subinterval $J \subseteq I$ such that $|f(t_1) - f(t_2)| \leq \varepsilon$ for all $t_1, t_2 \in J$.*

Proof. Let $F_n := \{t \in I \mid |f_n(t) - f_m(t)| \leq \frac{\varepsilon}{3}$ for all $m \geq n\}$. Obviously, every F_n is closed and $I = \cup F_n$. By Baire category theorem, at least one of them, say F_k contains a closed (why?) interval I'. For all $t \in I'$ and $m \geq k$ we have

$$|f_k(t) - f_m(t)| \leq \frac{\varepsilon}{3}$$

and hence also $|f_k(t) - f(t)| \leq \frac{\varepsilon}{3}$. Since f_k is uniformly continuous on I', there exists a closed subinterval J of I' so that

$$|f_k(s) - f_k(t)| \leq \frac{\varepsilon}{3} \text{ for all } s, t \in J.$$

For such points $s, t \in J$, we have

$$\begin{aligned} |f(s) - f(t)| &\leq |f(s) - f_k(t)| + |f_k(s) - f_k(t)| + |f_k(t) - f(t)| \\ &\leq 3\frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

□

Proof. From the lemma, it follows that for every closed subinterval I of $[a,b]$, we can construct a sequence of closed intervals $I_n := [a_n, b_n]$ such that $I_n \subseteq I$, $a_n < a_{n+1} \leq b_{n+1} < b_n$, $b_n - a_n < \frac{1}{n}$ and $|f(s) - f(t)| \leq \frac{1}{n}$ for $s, t \in I_n$. By Cantor's intersection ("nested interval") theorem, there exists a common point $x \in \cap I_n$. At this point f is obviously continuous.

Thus, if t is an arbitrary point of $[a,b]$ and $\varepsilon > 0$, then $[t - \varepsilon, t + \varepsilon]$ contains a point x at which f is continuous. Hence the set of points of continuity of f is dense in $[a,b]$. □

Ex. 11. An amusing exercise: Let (x_n) be any sequence of real numbers. Show that the set $\{x \in \mathbb{R} : x \neq x_n, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$. Hence conclude that $\mathbb{R}$ is uncountable.

Show that $\mathbb{Q}$ cannot be written as the intersection of a countable family of open subsets of $\mathbb{R}$.

The next result is a beautiful application of Baire's theorem which uses both the versions! To put it in perspective, recall that the pointwise limit of a sequence of continuous functions need not be continuous while the uniform limit is. However, the poinwise limit cannot be too wild.

Theorem 12. *Let X be a complete metric space. Let $f_n \colon X \to \mathbb{R}$ be a sequence of continuous functions. Assume that there exists a function $f \colon X \to \mathbb{R}$ such that $f_n(x)$ converges to $f(x)$ for each $x \in X$. Then there exists a dense subset D of X such that each point of D is a point of continuity of f.*

Proof. Fix $\varepsilon > 0$. Define, for each $k \in \mathbb{N}$,

$$E_k(\varepsilon) := \{x \in X : |f_n(x) - f_m(x)| \leq 1/k, \text{ for all } m, n \geq n\}.$$

Then we claim that $E_k(\varepsilon)$ is closed for each k.

> Reason: Fix $m, n \geq k$. Then the set $E_k^{m,n}(\varepsilon) := \{x \in X : |f_n(x) - f_m(x)| \leq 1/k\}$ is a closed subset of X, since $|f_n - f_m|$ is continuous. Now, since $E_k(\varepsilon) = \cap_{m,n\geq k} E_k^{m,n}(\varepsilon)$, the claim follows.

It is easy to show that $X = \cup_k E_k(\varepsilon)$.

> Reason: Let $x_0 \in X$. Since $f_n(x_0) \to f(x_0)$, the sequence $(f_n(x_0))$ is Cauchy. Hence for the given $\varepsilon > 0$, there exists k_0 such that for $m, n \geq k_0$, we have $|f_m(x_0) - f_n(x_0)| \leq 1/k_0$. Hence we conclude that $x_0 \in E_{k_0}(\varepsilon)$.

Since $X$is a complete metric space, at least one of $E_k(\varepsilon)$ should have nonempty interior. Let $U_\varepsilon := \cup_k \text{Int}\,(E_k(\varepsilon))$. Then U_ε is a nonempty open subset of X.

Let $U_n := U_{1/n}$. We claim that each U_n is dense in X.

> Reason: It is enough if we show that every closed ball $B := B[x, r]$ meets U_n nontrivially. (Why?)
>
> > Reason: To show a set A is dense in a metric space, it suffices to show that $A \cap B(x, r) \neq \emptyset$ for any $x \in X$and $r > 0$. Assume that $A \cap B[z, \rho] \neq \emptyset$ for any $z \in X$ and $\rho > 0$. Then given any $B(x, r)$, we may take $z = x$ and $\rho = r/2$. Then $\emptyset \neq A \cap B[x, \rho] \subset A \cap B(x, r)$.
>
> Observe that the closed set (and hence a complete metric space) B is the union of a countable family of closed sets: $B = \cup_n (B \cap E_k(1/n))$. By Baire, at least one of them has nonempty interior, say, $\text{Int}\,(B \cap E_k(1/n)) \neq \emptyset$. Since $\text{Int}\,(B \cap E_k(1/n)) \subset B \cap \text{Int}\, E_k(1/n)$, it follows that $B[x, r] \cap U_n \neq \emptyset$ and hence the claim is proved.

Let $D := \cap_n U_n$. By Baire, D is dense in X. We claim that every $x \in D$ is a point of continuity of f.

Reason: Fix $p \in D$. Let $\varepsilon > 0$ be given. Choose $N \gg 0$ such that $1/N < \varepsilon$. Since $p \in D$, $p \in U_N$ and hence there exists $k \in \mathbb{N}$ such that $p \in \text{Int}\,(E_k(1/N)$. By continuity of f_k at p, there exists an open neighbourhood V of p contained in $\text{Int}\, E_k(1/N)$ such that

$$|f_k(x) - f_k(p)| < \varepsilon, \text{ for all } x \in V. \tag{1}$$

For $x \in V$, since $V \subset E_k(1/N)$, by the definition of $E_k(\varepsilon)$'s, we have

$$|f_m(x) - f_k(x)| \leq 1/N, \text{ for all } m \geq k. \tag{2}$$

Letting $m \to \infty$ in the above equation, we obtain

$$|f(x) - f_k(x)| \leq 1/N, \text{ for all } x \in V. \tag{3}$$

We are now ready for the kill. We claim that $|f(x) - f(p)| < 3\varepsilon$ for $x \in V$.

$$\begin{aligned} |f(x) - f(p)| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(p)| + |f_k(p) - f(p)| \\ &\leq 1/N + \varepsilon + 1/N \\ &< 3\varepsilon. \end{aligned}$$

This shows that f is continuous at every point of D.

□

Theorem 13. *There exists a continuous function on* $[0, 1]$ *which is not differentiable at any point.*

Proof. For each positive integer n, we let

$$C_n := \left\{ f \in \mathcal{C}[0,1]] \;\middle|\; \left|\frac{f(t+h)-f(t)}{h}\right| \leq n \text{ for some } t \text{ and all } h \text{ with } t+h \in [0,1] \right\}.$$

We shall show that C_n is nowhere dense for each n. Since $\mathcal{C}[0,1]$ is a complete metric space, it will then follow that $\cup_{i=1}^{\infty} C_n \neq \mathcal{C}[0,1]$. Hence there exists $f \in \mathcal{C}[0,1]$ such that $f \notin C_n$ for any n.

This f is the required one. For, $f \notin C_n$ means that $\left|\frac{f(t+h)-f(t)}{h}\right| > n$ for all $t \in [0,1]$ and some h (depending on t and n). Note that for each fixed t, $h \to 0$ as $n \to \infty$: Because, given $\varepsilon > 0$, the difference quotient $\left|\frac{f(t+h)-f(t)}{h}\right|$ is bounded as a function of h for $|h| \geq \varepsilon$. Thus $\limsup_{h \to 0} \left|\frac{f(t+h)-f(t)}{h}\right| = \infty$, and the derivative of f at t fails to exist for each $t \in [0,1]$.

To prove that C_n is nowhere dense, we must show that $\overline{C_n}$ contains no non-empty open set. First we show that C_n is closed, i.e., $\overline{C_n} = C_n$.

Since $\mathcal{C}[0,1]$ is a metric space, it is first countable and hence it suffices to show that if $\{f_k\} \subseteq C_n$, with $\lim f_k = f \in \mathcal{C}[0,1]$, then $f \in C_n$. Now, $f_k \in C_n$ implies that there exists $t_k \in [0,1]$ such that

$$\left|\frac{f_k(t_k + h) - f_k(t_k)}{h}\right| \leq n, \quad \text{for all } h.$$

Since $[0,1]$ is compact, the sequence t_k has a convergent subsequence. We call this subsequence again by t_k and let $t_0 = \lim t_k$. Then

$$\begin{aligned}
\left|\frac{f(t_0+h)-f(t_0)}{h}\right| &\leq \left|\frac{f(t_0+h)-f(t_k+h)}{h}\right| + \left|\frac{f(t_k+h)-f_k(t_k+h)}{h}\right| \\
&\quad + \left|\frac{f_k(t_k+h)-f_k(t_k)}{h}\right| + \left|\frac{f_k(t_k)-f(t_k)}{h}\right| \\
&\quad + \left|\frac{f(t_k+h)-f(t_0)}{h}\right| \\
&= (1)+(2)+(3)+(4)+(5)
\end{aligned}$$

Now, fix h. For any $\varepsilon > 0$, if k is large enough, (1) and (5) are smaller than ε, since f is continuous, and $t_k \to t_0$. (2) and (4) are smaller than ε, since f_k converges uniformly to f. $(3) \leq n$. Hence, we get

$$\left|\frac{f(t_0+h)-f(t_0)}{h}\right| \leq n + 4\varepsilon \quad \text{for any } \varepsilon > 0$$

so that

$$\left|\frac{f(t_0+h)-f(t_0)}{h}\right| \leq n$$

and hence $f \in C_n$. Thus each C_n is closed.

We now show that C_n is nowhere dense; that is, given any $g \in C_n = \overline{C_n}$ and any $\varepsilon > 0$, there exists $f \in \mathcal{C}[0,1]$ such that $\| f - g \| < \varepsilon$ and $f \notin C_n$.

Now a typical example of a function in $\mathcal{C}[0,1]$ which is not in C_n is the "sawtooth" function. For any n, we can find such a function, whose norm is less than or equal to any prescribed $\varepsilon > 0$, and where the slope of each line segment is greater than n in absolute value. To find such a function $f \notin C_n$, with an ε-distance from g, we need only construct a sawtooth function close to g, as in Figure 1.

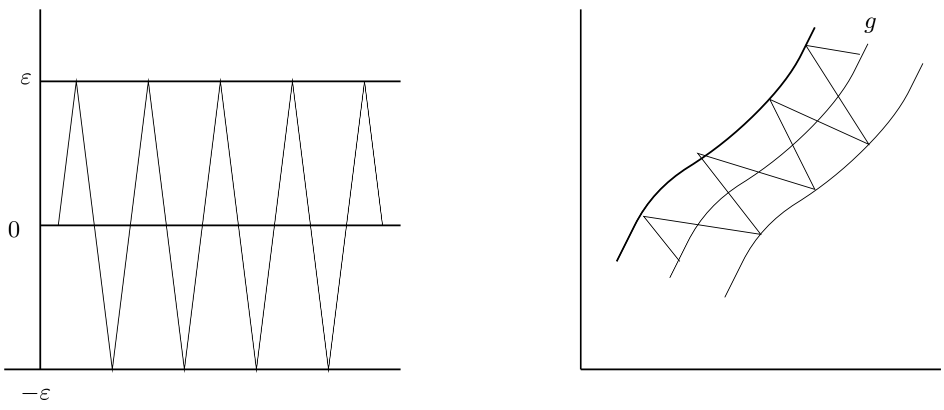

Figure 1: "Saw-tooth" Function

To construct such a function f, we use the uniform continuity of g to get a piecewise linear function g_1 with $\| g - g_1 \| < \frac{\varepsilon}{2}$. For each linear piece of g_1 we construct a saw-tooth function. $\square$

3 A Property of Repeated Integrals

Theorem 14. *Let $f \in \mathcal{C}[0,1]$. Let f_1 be any integral of f and let f_{n+1} be any integral of f_k, $k \geq 1$. If some f_k vanishes identically, so does f.*

We need only differentiate f_{k+1} repeatedly. We now wish to prove a generalization.

Theorem 15. *If, for each x, there is an integer $k = k(x)$, such that $f_k(x) = 0$, then f vanishes identically.*

Proof. Let $E_k := \{x \in [0,1] \mid f_k(x) = 0\}$. By hypothesis, any $x \in [0,1]$ lies in some E_k. E_k is obviously closed. Hence by Baire category theorem, there exists some k for which $\overline{E_k}$ contains an interval I_k. For this k, since f_k is continuous and vanishes on E_k, we have $f_k(x) = 0$ for all $x \in I_k$.

If $I_k \neq [0,1]$, we repeat this argument with any complementary subinterval and so on. In this way, we have $f(x) = 0$ for all points x of an everywhere dense set. Since f is continuous, $f = 0$ on $[0,1]$. □

An Alternate Proof. Let $I = [0,1]$. Let J_0 be a non-empty, closed subinterval such that $f_k(x) = 0$ for all $x \in J_0$. Then we have $f \equiv 0$ on J_0.

Let $E := \{x \in I \mid f(x) = 0\} \supseteq I_0$. E is closed. If $E \neq [0,1]$, then there exists an open interval in $J \setminus E$ and hence a non-empty closed subinterval $J \subseteq I \setminus E$. Note that $f(x) \neq 0$ for all $x \in J$.

Using the argument by replacing the pair (f, I) with (f, J), we conclude that there exists a non-empty closed subinterval $J_0 \subseteq J$ such that $f_l(x) = 0$ for all $x \in J_0$. Hence $f \equiv 0$ on J_0. This contradicts the fact that $J_0 \subseteq J \subseteq I \setminus E$ and hence $E = I$. □

4 A Characterization of Polynomials.

Let f on $[0,1]$ be such that its n-th derivative is 0. By repeated application of mean value theorem, we see that f coincides on $[0,1]$ with a polynomial (of degree atmost $n-1$). We generalize this as follows:

Theorem 16. *Let f be C^∞. Let, for each $x \in [0,1]$, there be $n(x) \in \mathbb{N}$ such that $f^{(n(x))}(x) = 0$. Then f coincides with a polynomial.*

Proof. Let E_n be the set of points x for which $f^{(n)}(x) = 0$. Then $[0,1] = \cup E_n$, by hypothesis. As E_n is closed, by Baire's theorem, there is a closed interval I in which some E_n is dense. Since $f^{(n)}$ is continuous, $f^{(n)} \equiv 0$ on I and f coincides with a polynomial on I. If $I = [0,1]$, we repeat the same reasoning in the remaining part of $[0,1]$ and so on. In this way, we see that there is a dense set of intervals in each of which f coincides with a polynomial. What needs to be shown is that f coincides with the same polynomial in all the intervals.

We are going to apply Baire's theorem again to the nowhere dense set H that is left when we remove the interiors of the dense set of intervals from $[0,1]$.

We show that H is perfect. H is closed, since it is obtained by removing a collection of open intervals from a closed interval. If H is empty, then there was only one interval to begin with and there is nothing to prove then. So we assume H is non-empty. If H were not perfect, then H contains a point y that is not a limit point. y is then the common end point of two intervals in each of which f coincides with some polynomial. Then if n exceeds the degrees of these two polynomials, then $f^{(n)}(x) = 0$ for x in both intervals, and also at the common end point by continuity. Therefore, f coincides with a polynomial in the union of the two intervals. But then $y \notin H$, a contradiction.

It remains to be shown that $H \neq \emptyset$ leads to a contradiction.

If $H \neq \emptyset$, H is closed and hence a complete metric space. Now the J has intervals K complementary to H, i.e., "discarded" in the first part of the proof. On such K, we have $f^{(m)}(x) = 0$, $m = m(k)$, for all $x \in K$. If $m \leq k$, $f^{(n)}(x) = 0$, in K by differentiation. If $m > k$, $f^{(k)}(x) = f^{(k+1)}(x) = \cdots = 0$ at the end points of K, since these are points of H. If we integrate $f^{(m)}$ repeatedly, we get $f^{(k)}(x) = 0$ throughout K.

This reasoning is applicable to all intervals K which are complementary to H and lie in J. Thus $f^{(k)}(x) = 0$ for all points of J. But then $J \cap H = \emptyset$. This contradiction shows that H must be empty. This implies that there was only one interval which was "thrown out". □

Theorem 17. *Let $f : [0, \infty) \to [0, \infty)$ be continuous. Assume that $\lim_{n\to\infty} f(nx) = 0$, for any $x \in [0, \infty)$. Then $\lim_{x\to\infty} f(x)$ exists and is zero.*

Proof. If the limit $\lim_{n\to\infty} f(x)$ exists, one can easily show that it has to be 0. So, the main thrust of the claim is that the limit exists. Let $\epsilon > 0$ be given.

For each n, let $E_n := \{x \in [1, \infty) \mid |f(mx)| \leq \epsilon \text{ for all } m \geq n\}$. Then each E_n is closed subset of $[0, \infty)$. By hypothesis, $[1, \infty) = \cup E_n$. Since $[1, \infty)$ is complete, Baire's theorem implies the existence of n for which E_n contains an open interval, say, (a, b), $a < b$. By our definition of E_n, we have

$$|f(kx)| < \epsilon \text{ for } \textit{all } x \in (a, b) \text{ and } k \geq n.$$

If k is sufficiently large, then $(ka, kb) \cap ((k + 1)a, (k + 1)b) \neq 0$. That is, we choose $k \geq n$ with $(k + 1)a < kb$ or $a < k(b - a)$ or $k > \max\{n, \frac{a}{b-a}\}$. Then by hypothesis, on the union of all such intervals, we have $|f(x)| < \epsilon$. But then their union is of the form (ka, ∞). Then for $x \geq ka$, we have $|f(x)| < \epsilon$. Hence $\lim_{x\to\infty} f(x)$ exists and is 0. □

Theorem 18. *The set of continuous nowhere monotonic ($\equiv$ every where oscillating) functions is dense in $\mathcal{C}[0, 1]$.*

Proof. First notice that it is easy to give discontinuous nowhere monotonic functions.

Let $\{I_n\}$ be the sequence of *all* closed intervals of $[0, 1]$ with rational end points. Let G_n denote the subset of $\mathcal{C}[0, 1]$ of functions which are not monotonic on I_n. (Read the description of G_n carefully! We are not saying "nowhere monotonic on I_n".) The idea is to show that G_n is both open and dense in $\mathcal{C}[0, 1]$ and apply Baire's theorem.

Claim: G_n is open.

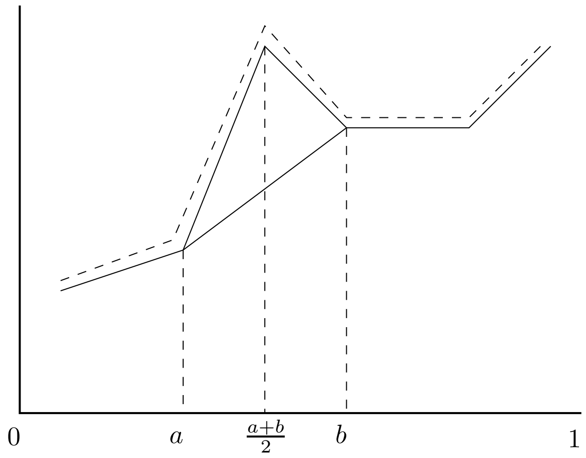

Figure 2:

If $f \in G_n$, f is not monotonic on I_n. Hence there exists $x < y < z$ with $f(x) < f(y)$ but $f(z) < f(y)$ (or $f(x) > f(y)$ and $f(y) < f(z)$). Now if we choose g with $\|f - g\| < \frac{1}{2}\min\{f(y) - f(x), f(y) - f(z)\}$, we must have g must have similar properties. Thus all elements g which are near to f lie in G_n.

Claim: G_n is dense, i.e., every neighborhood contains an f which is not monotonic on I_n.

It is enough to prove that G_n is dense in $PL[0,1]$, the space of piecewise linear functions, since $PL[0,1]$ is dense in $\mathcal{C}[0,1]$.

Let $f \in PL[0,1] \setminus G_n$. Then f is piecewise linear and monotonic on I_n. We may assume that f is non-decreasing on I_n.

Let $[a,b] \subseteq I_n$ on which f is linear. If m is the slope of this line segment, the graph of f on $[a,b]$, we define

$$h(x) = \begin{cases} 0 & 0 \leq x \leq a \\ 2m(x-a) & a < x \leq \frac{a+b}{2} \\ -2m(x-b) & \frac{a+b}{2} < x \leq b \\ 0 & b \leq x < 1 \end{cases}$$

which is in $PL[0,1]$. Let $g = f + h$. Then $g(\frac{a+b}{2}) > g(a), g(b)$. But however, $|g(x) - f(x)| = |h(x)| < m(b-a)$. Hence if $\epsilon > 0$ is given, the subinterval $[a,b]$ may be chosen so that $m(b-a) < \epsilon$. Hence $g \in B(f, \epsilon)$. Thus G_n is dense in $PL[0,1]$ and hence in $\mathcal{C}[0,1]$.

Since $\{G_n\}$ are both open and dense in $\mathcal{C}[0,1]$, $G = \cap G_n$ is also dense in $\mathcal{C}[0,1]$ and therefore cannot be empty. However, if $f \in G$, then $f \in G_n$ for all n and hence f is not monotonic on any subinterval of $[0,1]$ with rational end points. The result follows. □

Definition 19. Oscillation of a function:

Theorem 20. *Let $f\colon (a,b) \to \mathbb{R}$ be given. Then*
(i) *f is continuous at $c \in (a,b)$ iff $\omega_f(c) = 0$.*
(ii) *For any $\varepsilon > 0$, the set $\{x \in (a,b) : \omega_f(x) < \varepsilon\}$ is open.*

Proof. Let f be continuous at c. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that for $x \in J := (x - \delta, x + \delta)$, we have $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$ so that for $x, y \in J$, we have

$$f(y) - f(x) \leq f(c) + \varepsilon - (f(c) - \varepsilon) = 2\varepsilon.$$

In particular, $\omega_f(J) < 2\varepsilon$. It follows that

$$0 \leq \omega_f(c) \leq \inf\{\omega_f(J) : \text{ where } J \text{ corresponds to } \varepsilon \text{ as above}\} = 0.$$

Conversely, if $\omega_f(c) = 0$, then given $\varepsilon > 0$, there exists an open interval $J \ni c$ such that $\omega_f(J) < \varepsilon$. Choose $\delta > 0$ such that $(c - \delta, c + \delta) \subset J$. In particular, for $x \in (c - \delta, c + \delta)$, we have $|f(x) - f(c)| < \varepsilon$. Thus, f is continuous at c. This proves (i).

To prove (ii), let $\varepsilon > 0$ be given. Let c be such that $\omega_f(c) < \varepsilon$. Thus, there exists an open interval $J \ni c$ such that $\omega_f(J) < \varepsilon$. Thus for any $x \in J$, we have $\omega_f(x) \leq \omega_f(J) < \varepsilon$. In other words, $c \in J \subset \{y : \omega_f(y) < \varepsilon\}$. Thus c is an interior point of the set under question and so the set is open. □

Theorem 21. *There exists no function $f \colon (0, 1) \to \mathbb{R}$ which is continuous at rationals and discontinuous at irrationals.*

Proof. Assume such a function f exists. Consider the sets

$$G_n := \{x \in (0, 1) : \omega_f(x) < 1/n\}.$$

These sets are open in $(0, 1)$ by the last lemma. Since $(0, 1)$ is open in $[0, 1]$, G_n is open in $[0, 1]$. Since it contains $\mathbb{Q} \cap (0, 1)$, it is dense in $[0, 1]$. Hence its complement $F_n := [0, 1] \setminus G_n$ in $[0, 1]$ is nowhere dense and closed. Since $\cap_n G_n = (0, 1) \cap \mathbb{Q}$, it follows that $\cup_n F_n$ is the set of irrationals union $\{0, 1\}$. If we enumerate the set $(0, 1) \cap \mathbb{Q}$ as $\{r_n : n \in \mathbb{N}\}$, it follows that

$$[0, 1] = \cup_n F_n \cup \{r_1\} \cup \cdots \{r_n\} \cup \cdots,$$

is a countable union of nowhere dense sets. This contradicts Baire Category theorem. □

Theorem 22 (Baire category theorem for locally compact spaces)**.** *Let X be a locally compact hausdorff space. Let (U_n) be a sequence of open dense sets in X. Then $\cap_n U_n$ is dense in X.*

Proof. Let G be a nonempty open set in X. We need to prove that there exists $x \in G$ such that $x \in U_n$ for all n. The strategy is to mimic the proof in the case of metric spaces replacing open balls by the existence of open sets V such that $\overline{V}$ is compact and $x \in V \subset \overline{V} \subset U$ for any given open set U and $x \in U$ and then invoking Cantor intersection theorem for a decreasing sequence of compact sets.

Since G is a nonempty open set and U_1 is dense, there exists $x_1 \in G \cap U_1$. Since $G \cap U_1$ is open, $x \in G \cap U_1$ and X is locally compact hausdorff space, there exists an open set V_1 such that $x \in V_1$, $\overline{V}_1$ is compact and $\overline{V}_1 \subset G \cap U_1$. Assume, by way of induction, that we have chosen x_i, $V_i \ni x_i$, $\overline{V}_i$ is compact and that $x_i \in V_i \subset \overline{V}_i \subset V_{i-1} \cap U_i$, for $1 \leq i \leq n$.

Now given a nonempty open set V_n, since $V_n \cap U_{n+1}$ is nonempty, there exists $x_{n+1} \in V_n \cap U_{n+1}$. Since X is locally compact and hausdorff, there exists an open set $V_{n+1} \ni x_{n+1}$

such that $\overline{V}_{n+1}$ is compact and $x_{n+1} \in V_{n+1} \subset \overline{V}_{n+1} \subset V_n \cap U_{n+1}$. Let $K_n := \overline{V}_n$. Thus we have a decreasing sequence (K_n) of nonempty compact subsets. Hence by Cantor intersection theorem, there exists $x \in \cap_n K_n$. Since $x \in K_n = \overline{V}_n \subset U_n$, it follows that $x \in \cap U_n$. Also, $x \in K_1 \subset U$. □